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Solution of a Homogeneous Linear Recurrence Relation

Solución de una Relación de Recurrencia Lineal y Homogénea

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Abstract

We exhibit the Baldoni et al method [1] to solve a homogeneous linear recurrence relation, with applications to Chebyshev polynomials and Fibonacci numbers.

***Keywords*:** Recurrence relations, Chebyshev polynomials, Baldoni's algorithm, Fibonacci numbers.

Resumen

Se muestra el método de Baldoni et al [1] para resolver una relación de recurrencia lineal y homogénea, con aplicaciones a los polinomios de Chebyshev y a los números de Fibonacci.

***Palabras Claves*:** Relaciones de recurrencia, polinomios de Chebyshev, algoritmo de Baldoni, números de Fibonacci .

1. Introduction

Here we consider recurrence relations with the structure:

$$a_{n+k} = b_{k-1}a_{n+k-1} + b_{k-2}a_{n+k-2} + \ldots + b_0 a_n, \quad (1)$$

$n, k \geq 0$, and the initial values $a_0, a_1, ..., a_{k-1}$. In according with the Baldoni et al process [1], first we construct the companion matrix [2,3]:

$$A_{kxk} = \begin{pmatrix} 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & & 0 \\ & & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ b_0 & b_1 & b_2 & b_3 & & b_{k-1} \end{pmatrix} \quad (2)$$

whose characteristic equation [4] is:

$$\lambda^k - b_{k-1}\lambda^{k-1} - b_{k-2}\lambda^{k-2} - \cdots - b_1\lambda - b_0 = 0, \quad (3)$$

and we accept that it has distinct roots. Therefore, the solution of 1 is given by:

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_k\lambda_k^n, \quad n = k, k+1, \cdots \quad (4)$$

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where the c_j are determined through the initial values via the linear system:

$$c_1\lambda_1^r + c_2\lambda_2^r + \cdots + c_k\lambda_k^r = a_r, \qquad (5)$$

$r = 0, 1, \cdots, k-1$. The Sec. 2 has applications of this procedure to Chebyshev polynomials [5,6,7] and Fibonacci numbers [8,9,10].

2. Some Applications of Baldoni et al Method

The Chebyshev-Lanczos polynomials $T_n(x)$ are defined by [6,7]:

$$T_{n+2} = 2xT_{n+1} - T_n \quad T_0 = 1, \quad T_1 = x, \qquad (6)$$

$x\epsilon[-1, 1]$.

then 1 gives $k = 2$, $b_0 = -1$, $b_1 = 2x$, thus from **??**ecu2), 3 and 5

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 2x \end{pmatrix} \qquad \lambda^2 - 2x\lambda + 1 = 0$$

$$\lambda_1 = x + \sqrt{x^2-1}, \qquad \lambda_2 = x - \sqrt{x^2-1} \qquad (7)$$

$$c_1 + c_2 = 1, \quad c_1\lambda_1 + c_2\lambda_2 = x \quad \therefore \quad c_1 = c_2 = \frac{1}{2},$$

and from 4 we obtain the solution of 6:

$$a_n = \frac{1}{2}\left(\lambda_1^n + \lambda_2^n\right) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^2 + \left(x - i\sqrt{1-x^2}\right)^2 = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right),\right. \quad x = \cos\theta,$$

therefore $T_n(x) = \cos n\theta$, which is fundamental in problems of interpolation [11] and in the tau method of Lanczos-Ortiz [11,12]. The Chebyshev polynomials of second kind $U_n(x)$ [7] satisfy the recurrence relation 6 but with different initial conditions:

$$U_{n+2} = 2xU_{n+1} - U_n, \quad U_0 = 1, \quad U_1 = 2x, \qquad (8)$$

$x\epsilon\,[1-, 1]$, then the corresponding eigenvalues are in 7 and now the linear system 5 is:

$$c_1 + c_2 = 1, \qquad c_1\lambda_1 + c_2\lambda_2 = 2x$$

$$\therefore$$

$$c_1 = \frac{\lambda_1}{2\sqrt{x^2-1}}, \qquad c_2 = -\frac{\lambda_2}{2\sqrt{x^2-1}}$$

,

and 4 gives the solution:

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{x^2-1}}\left(\lambda_1^{n+1} + \lambda_2^{n+1}\right) \\ &= \frac{1}{2i\sqrt{1-x^2}}\left[\left(x + i\sqrt{1-x^2}\right)^{n+1} - \left(x - i\sqrt{1-x^2}\right)^{n+1}\right. \\ &= \frac{1}{2i\sin\theta}\left(e^{i(n+1)\theta} + e^{-i(n+1)\theta}\right), \end{aligned}$$

$\therefore U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, in according with the literature [6,7].

A similar process can be applied to Chebyshev polynomials of third kind $V_n(x)$ [7] and fourth kind $W_n(x)$ [7,13,14] verifying the recurrence relation (6) with the initial values $V_0 = 1, V_1 = 2x - 1, W_0 = 1, W_1 = 2x + 1$ to deduce the important expressions:

$$V_n(x) = \frac{\cos\left(n + \frac{1}{2}\right)\theta}{\cos\left(\frac{\theta}{2}\right)},$$

$$W_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}, \qquad x = \cos\theta. \qquad (9)$$

The Fibonacci numbers F_n [8] satisfy the recurrence relation [9,10]:

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \qquad F_1 = 1, \qquad (10)$$

then 1 and 3 give $k = 2, b_0 = b_1 = 1$ and $\lambda^2 - \lambda - 1 = 0$, thus:

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_1 = \frac{1-\sqrt{5}}{2}, \quad c_1 + c_2 = 0,$$
$$c_1\lambda_1 + c_2\lambda_2 = 1$$

,

therefore $c_1 = -c_2 = \frac{1}{\sqrt{5}}$ and (4) implies the Binet?s formula:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right], \qquad (11)$$

with $n = 0, 1, 1, \cdots$ The golden ratio $\frac{1+\sqrt{5}}{2}$ is sometimes denoted by the letter Φ, from the name of the Greek artist Phidias who often used this ratio in his sculptures. It is possible to write 11 in terms of the hypergeometric function [15,16]:

$$F_n = \frac{n}{2^{n-1}}\,{}_2F_1\left(\frac{1-n}{2}, \frac{2-n}{2}; \frac{3}{2}; 5\right). \qquad (12)$$

The Lucas number L_n [10,17] verify the recurrence relation 10 with the initial values $L_0 = 2, L_1 = 1$, then this Baldoni?s algorithm leads to:

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right). \qquad (13)$$

Conclusions

Baldoni et al approach[1] is powerful to solve homogeneous linear recurrence relations, for example, here it showed that 6 gives the four types of Chebyshev polynomials [7] if we employ different initial values. Similarly, 10 implies the Fibonacci and Lucas numbers for distinct initial conditions.

References

[1] M. W. Baldoni, C. Ciliberto, G. M. Piacentini Cattaneo, Elementary number theory, cryptography and codes, Springer, Berlin, 2009.
[2] L. Brand, The companion matrix and its properties, Am. Math. Monthly 71, No. 6, 629-634, 1964.
[3] J. H. Wilkinson, The algebraic eigenvalue problem, Clarendon Press, Oxford, 1965.
[4] I. Guerrero, J. López-Bonilla, J. Rivera, Leverrier-Takeno coefficients for the characteristic polynomial of a matrix, J. Inst. Eng. (Nepal) 8, No. 1-2, 255-258, 2011.
[5] P. L. Chebyshev, Théorie des mécanismes connus sous le nom de parallélogrammes, Mém. Acad. Sci. Pétersb. 7, 539-568, 1854.
[6] C. Lanczos, Applied analysis, Dover, New York, 1988.
[7] J. C. Mason, D. C. Handscomb, Chebyshev polynomials, Chapman & Hall/CRC, London, 2002.
[8] L. Pisano (Fibonacci), Liber Abaci, 1202.
[9] R. A. Dunlap, The golden ratio and Fibonacci numbers, World Scientific, Singapore, 1997.
[10] S. Vajda, Fibonacci and Lucas numbers and the golden ratio section: Theory and applications, Dover, New York, 2008.
[11] C. Lanczos, Trigonometric interpolation of empirical and analytical functions, J. Math. Phys. 17, 123-199, 1938.
[12] J. H. Caltenco, B. Moreno L., J. López-Bonilla, J. Morales, Sobre el método tau de Lanczos-Ortiz, Journal de Ciencia e Ingeniería 5, No. 1, 20-24, 2013.
[13] W. Gautschi, On mean convergence of extended Lagrange interpolation, J. Compt. Appl. Math. 43, 19-35, 1992.
[14] L. Boyadjiev, R. Scherer, On the Chebyshev polynomials, Kuwait J. Sci. Eng. 28, No. 2, 227-240, 2001.
[15] K.Dilcher, Hypergeometric functions and Fibonacci numbers, Fibonacci Quart. 38, No.4, 342-363, 2000.
[16] J. Patricia Hannah, Identities for the gamma and hypergeometric functions: an overview from Euler to the present, Master of Science Thesis, Univ. of the Witwatersrand, Johannesburg, South Africa, 2013.
[17] E. Lucas, Théorie des nombres, Gauthier-Villars, 1891.