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Another Extension of the Little q-Jacobi Polynomial and its Properties

Otra Extensión del q-Polinomio de Jacobi y sus Propiedades

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Abstract

The little q-Jacobi polynomial is extended in this work whose properties such as fractional q-integral, q-integral representations, q-derivative, higher order q-derivative, q-Liebnitz's formula, summation formula are obtained. Also, the Fatou's lemma of measurable function is applied on this polynomial.

***Keywords*:** q-polynomial, q-fractional integral, q-derivative, q-fractional derivative, Lebesgue measure.

Resumen

En este trabajo se generaliza el q-polinomio de Jacobi y se obtienen sus propiedades q-integral fraccional, q-representaciones integrales, q-derivada, q-derivadas de alto orden, q-fórmula de Leibnitz y la fórmula de suma. Además, el lema de Fatou de función medible es aplicado a este polinomio.

***Palabras Claves*:** q-polinomio, q-integral fraccional, q-derivada, q-derivada fraccional, medida de lebesgue.

1. Introducción

The little q-Jacobi polynomial is given by ([2], p.161, Eq.(7.1.11))

$$p_n(x\,;\,a,\,b\,;\,q) = {}_2\varphi_1\left[\begin{matrix} q^{-n},\; abq^{n+1};\; q,\; xq \\ aq; \end{matrix}\right], \tag{1}$$

we consider an extension of this polynomial in the form

$$p_n^r(x\,;s,\;a,\,b\,;\,q) = \sum_{k=0}^{[n/s]} \frac{(\,q^{-n}\,;\,q)_{sk}\,(\,abq^{n+1}\,;\,q)_{sk}}{(\,aq\,;\,q)_{rk}\,(\,q\,;\,q)_k} x^k q^k \tag{2}$$

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where $|q| < 1$, and n and s are positive integers, r is any integer, and a and b are complex numbers. For this polynomial, we have obtained fractional q-integral formula, q-integral representations, q-derivative expression, higher order q-derivative, q-Liebnitz's formula, a summation formula and a result based on Lebesgue measure over $\mathbb{R}$.

The following formulas and definitions are used in this paper [2].

The q-Factorial function is defined by,

$$(a\,;q)_n = \begin{cases} 1\,, \;\; if\; n=0 \\ (1-a)(1-aq^2)\ldots(1-aq^{n-1})\,, \;\; if\; n=1,\; 2,\; 3,\; \ldots \\ \frac{(a\,;q)_\infty}{(aq^n\,;q)_\infty}, \;\; if\; n\in\mathbb{C} \end{cases} \tag{3}$$

The q-Beta function is given by

$$(1.4) \quad B_q(x,y) = \int_0^1 \frac{t^{x-1}\,(tq;q)_\infty}{(tq^y;q)_\infty}\, d_qt \, = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \tag{4}$$

The Kober fractional integral formula ([6], p.187, Eq.(1.1)), which is a generalization of the Riemann-Liouville fractional integral formula ([6], p.187, Eq.(1.1)), is defined by

$$I_q^{\eta,\,\mu} f(x) \; = \; \frac{x^{-\,\eta\,-\,\mu}}{\Gamma_q(\mu)} \int_0^x (x - t\,q)_{\mu\,-\,1} t^\eta f(t)\; d_qt, \tag{5}$$

where $(x-y)_n = \; x^n(y/x\,;\,q)_n,\; \mathrm{Re}(\,\mu\,)\, > 0,\; \eta\in\mathbb{R}$ and $|q| < 1$.

Andrews and Askey's integral is given by ([1], p.98, Eq.(1.12))

$$\int_0^\infty t^{\alpha-1} \frac{(-ctq^{\alpha+\beta};q)_\infty}{(-ct;q)_\infty}\, d_qt = \frac{\Gamma_q(\alpha)\;\Gamma_q(\beta)\;(-cq^\alpha, -q^{1-\alpha}/c;q)_\infty}{\Gamma_q(\alpha+\beta)\;(-c,-q/c;q)_\infty}, \quad \mathrm{Re}(\alpha,\;\beta)\; > \;0 \tag{6}$$

W. Hahn's q-integral formula ([3], p.10, Eq,(3.16)) is defined by

$$\int_0^\infty t^{\lambda-1}\; e_q(-t)\; d_qt = \frac{(1-q)\;(q;q)_\infty}{(q^\lambda;q)_\infty} q^{-\lambda(\lambda-1)/2}, \tag{7}$$

where $\mathrm{Re}(\lambda) > 0$.

Here, we find the q-derivatives of the polynomial and write the expression for a general $\left(m^{th}\right)$ order. Now, compared with a general formula

$$D_q^m f_n(x) = \frac{1}{x^m(q-1)^m q^{m(m-1)/2}} \sum_{j=0}^m (-1)^j q^{j(j-1)/2} \begin{bmatrix} m \\ j \end{bmatrix}_q f_n(xq^{m-j}) \tag{8}$$

this is due to D. S. Moak ([4], p.33, Eq.(6.9)), we obtained a particular summation formula connecting degree extended little q-Jacobi polynomial 2 with the sum of its lower degrees. The formula 8 is also known as q-Leibnitz's formula. The following are the results obtained in this paper. On applying the Kober fractional integral formula 5 ([6], p.187, Eq.(1.1)) to the q-polynomial (1.2), we have

$$I_q^{\,\eta,\,\mu}\{p_n^r(x\,;s,\;\;a,\;b\,;\;q)\}\;\; = \frac{x^{-1}\,\Gamma_q(\mu-1)}{\Gamma_q(\mu)} \sum_{k=0}^{[n/s]} \frac{(\,q^{-n}\,;\;q)_{sk}\;(\,abq^{n+1}\,;\;q)_{sk}(\eta q\,;q)_k}{(\,aq\,;\;q)_{rk}\;(\,q\,;\;q)_k(\eta\mu\,;q)_k} q^k x^k$$

Next, the following q-integral representations are derived by making use of 6 and 7.

$$p_n^r(x\,;s,\ a,\ b\ ;\ q) = \frac{(1-q)(aq\ :q)_\infty(-c\ :q)_\infty(-q/c:q)_\infty}{(q\ :q)_\infty(-abcq^{n+1}\ :q)_\infty(-abq^n/c\ :q)_\infty(a^2b^2q^{n+1}\ :q)_\infty} \int_0^\infty t^{(a+b+n)} \frac{B_n^r(x,\ t^s;s,\ a,b,c;q)}{(-ct\,;q)_\infty(-a^2b^2ctq^{n+1}\,;q)_\infty}\ d_qt \tag{9}$$

where

$$B_n^r(x,\ t^s;s,\ a,b,c;q) = \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ a^2b^2q^{n+1}\ ;\ q)_{sk}\ (\ -abcq^{n+1}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k} x^kq^kt^{sk}$$

$$p_n^r(x\,;s,\ a,\ b\ ;\ q) = \frac{(abq^{n+1}\,;q)_\infty}{(1-q)(q\ ;q)_\infty} \int_0^\infty t^{(a+b+n)}\ e_q(-t)\,J_n^r(x,\ t^s;s\ ,\ a,\ b;\ q)\,d_qt\ , \tag{10}$$

where $J_n^r(x,\ t^s;s\ ,\ a,\ b;\ q) = \sum_{k=0}^{[n/s]} q^{(a+b+n+sk+1)(a+b+n+sk)/2} \frac{(\ q^{-n}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_k\ (\ q\ ;\ q)_k} x^kq^kt^{sk}$

Following formula is derived using 8 as

$$\left(\prod_{j=0}^{m-1} \frac{(q^{-n+sj};q)_s(abq^{n+sj+1};q)_s}{(aq^{jr+1};q)_r}\right) p_{n-ms}^r(x\ ;\ aq^{mr},\ bq^{m(2s-r)}\ ;\ q)\ = \frac{(-1)^m}{x^mq^{m(m-1)/2}}\sum_{i=0}^m (-1)^i q^{i(i-1)/2} \begin{bmatrix} m \\ i \end{bmatrix}_q p_{n-ms}^r(xq^{m-i};s\ ,\ a\ ,\ b\,;q) \tag{11}$$

where $ms \le n$.

Finally, by using the Fatou's lemma [[5], p.86] we arrived at,

"If $\{f_n\}_1^\infty$ is a sequence of non negative measurable functions and $f_n \to f$ almost everywhere on a set E , then $\int_E f\ \le\ \varliminf_{n\to\infty} \int_E f_n$ ".

Following inequality

$$\int_E V_r(x;\ s\ ,\ a\ ,\ b\,;q)\ dm\ \ \le\ \varliminf_{n\to\infty} \int_E p_n^r(xq^{sn}\ ;\ s,\ a,\ b\ ;\ q)\ dm, \tag{12}$$

is derived where $X = [\ -\infty, 0]$, $\mathfrak{M}$ = power set of X, $m\,(E)$ = lenght of E, $\forall E \in \Omega$, $x \in X$, $a > 0$, $b > 0$, r is an integer, s and n are positive integers, $0 < q\ < 1$, $V_r(x;\ s\ ,\ a\ ,\ b\,;q)$ is an absolutely convergent q-series (as a ${}_0\Phi_{r^r}$ - series) for every x which is given by

$$V_r(x;\ s\ ,\ a\ ,\ b\,;q)\ =\ \sum_{k=0}^\infty \frac{(-1)^k q^{sk(sk-1)/2}}{(aq\,;q)_{r\,k}(q\,;q)_k} x^kq^k\ = \lim_{n\to\infty} p_n^r(xq^n\ ;s,\ a,\ b\ ;\ q)$$

2. Proof of the Results

Using 2 and 5, we get

$$I_q^{n,\,\mu} p_n^r(x\,;s,\ a,\ b\ ;\ q)\ = \frac{x^{-\,n\,-\,\mu}}{\Gamma_q(\mu)} \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ abq^{n+1}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k} q^k \int_0^x (x-tq)_{\mu\,-\,1} t^{n+k}\ d_qt$$

$$= \frac{x^{-\ \eta-\ \mu}}{\Gamma_q(\mu)} \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ abq^{n+1}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k} q^k x^{\eta+k+\mu-1} \int_0^1 (yq)_{\mu\,-\,1} y^{\eta+k}\ \ d_q y \tag{13}$$

$$= \frac{x^{-1}\,\Gamma_q(\mu-1)}{\Gamma_q(\mu)} \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ abq^{n+1}\ ;\ q)_{sk} (\eta q\,;q)_k}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k (\eta\mu\,;q)_k} q^k x^k$$

Now, using 2 and 6, we get

$$p_n^r(x\,;s,\ \ a,\ \ b\ ;\ q) = \frac{(1-q)(aq\ :q)_\infty(-c\ :q)_\infty(-q/c\ :q)_\infty}{(q\ :q)_\infty(-abcq^{n+1}\ \ :q)_\infty(-abq^n/c\ \ :q)_\infty(a^2b^2q^{n+1}\ \ :q)_\infty}$$

$$\int_0^\infty t^{(\,a\,+\,b\,+\,n)} \frac{B_n^r(x,\ t^s; s,\ a, b, c; q)}{(-ct\,;q)_\infty(-a^2b^2ctq^{n+1}\,;q)_\infty}\ d_q t \tag{14}$$

where

$$B_n^r(x,\ t^s; s,\ a, b, c; q) = \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ a^2b^2q^{n+1}\ ;\ q)_{sk}\ (\ -abcq^{n+1}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k} x^k q^k t^{sk}$$

The following q-integral representation is obtained by making use of 2 and 7

$$p_n^r(x\,;s,\ \ a,\ \ b\ ;\ q) = \frac{(abq^{n+1}\,;q)_\infty}{(1-q)(q\ ;q)_\infty} \int_0^\infty t^{(\,a\,+\,b\,+\,n)}\ e_q(-\,t)\ J_n^r(x,\ t^s; s\,,\ \ a,\ \ b;\ \ q)\, d_q t \tag{15}$$

where

$$J_n^r(x,\ t^s; s\,,\ \ a,\ \ b;\ \ q) = \sum_{k=0}^{[n/s]} q^{(a+b+n+sk+1)(a+b+n+sk)/2} \frac{(\ q^{-n}\ ;\ q)_{sk}}{(\ aq\ ;\ q)_k\ (\ q\ ;\ q)_k} x^k q^k t^{sk}.$$

Next, the q-derivative operator is defined [2] as

$$D_q(f(x)) = \frac{f(xq)\ -\ f(x)}{xq\ -\ x}$$

The q-derivative of the polynomial 2 is

$$D_q\,\{p_n^r(x\,;s,\ \ a,\ \ b\ ;\ q)\} = \sum_{k=0}^{[n/s]} \frac{(\ q^{-n}\ ;\ q)_{sk}\ (\ abq^{n+1}\ ;\ q)_{sk} q^k}{(\ aq\ ;\ q)_{rk}\ (\ q\ ;\ q)_k} D_q x^k$$

$$= \frac{1}{(1-q)} \left(\frac{(q^{-n+s}; q)_s (abq^{n+s+1}; q)_s}{(aq; q)_r} \right) p_{n-s}(x\ ;\ aq^r,\ bq^{(2s-r)}\ ;\ q)$$

Similarly, second order q-derivative is given by

$$D_q^2\,\{p_n^r(x\,;s,\ \ a,\ \ b\ ;\ q)\} = \frac{1}{(1-q)^2} \left(\prod_{j=0}^{1} \frac{(q^{-n+sj}; q)_s (abq^{n+sj+1}; q)_s}{(aq^{jr+1}; q)_r} \right) p_{n-2s}(x\ ;\ aq^{2r},\ bq^{2(2s-r)}\ ;\ q)$$

In general, if $sm\ < -n$, then m^{th} order q-derivative is given by

$$D_q^m\left\{p_n^r(x\,;s,\ a,\ b\,;\ q)\right\}=\frac{1}{(1-q)^m}\left(\prod_{j=0}^{m-1}\frac{(q^{-n+sj};q)_s(abq^{n+sj+1};q)_s}{(aq^{jr+1};q)_r}\right)p_{n-ms}(x\,;\ aq^{mr},\ bq^{m(2s-r)}\,;\ q). \tag{16}$$

On making use of 8 and 16, the summation formula obtained as

$$\left(\prod_{j=0}^{m-1}\frac{(q^{-n+sj};q)_s(abq^{n+sj+1};q)_s}{(aq^{jr+1};q)_r}\right)p_{n-ms}^r\left(x\,;\ aq^{mr},\ bq^{m(2s-r)}\,;\ q\right)\ =$$

$$\frac{(-1)^m}{x^m q^{m(m-1)/2}}\sum_{i=0}^{m}(-1)^i q^{i(i-1)/2}\begin{bmatrix} m\\ i\end{bmatrix}_q p_{n-ms}^r\left(xq^{m-i};s\,,\ a\,,\ b\,;q\right) \tag{17}$$

We know that a polynomial is a continuous function and hence, is measurable function [5]. Thus the polynomial 2 is a measurable function. If $\{f_n(x)\}$ is sequence of measurable functions and converges to $f(x)$ then $f(x)$ is measurable function [5]. Now, we have

$$p_n^r(xq^n\,;s,\ a,\ b\,;\ q)\to V_r(x;\ s\,,\ a\,,\ b\,;q)\quad\text{as}\quad n\to\infty,$$

where $V_r(x;\ s\,,\ a\,,\ b\,;q)\ =\ \sum\limits_{k=0}^{\infty}\frac{(-1)^k q^{sk(sk-1)/2}}{(aq\,;q)_{r\,k}(q\,;q)_k}x^k q^k$

Now, consider the Lebesgue measure space $X, \mathfrak{M}, m, whereX, m(E)$ = length of $E, \forall E\in\Omega,\ x\in X,\ a>0, b>0$, r is an integer, s and n are positive integers, and $0<q<1$, then $\{p_n^r(x\,;s,\ a,\ b\,;\ q)\}_1^\infty$ becomes a sequence of non negative measurable functions. Hence, $\{p_n^r(xq^n\,;s,\ a,\ b\,;\ q)\}_1^\infty$ will also be a sequence of non negative measurable functions, and as $n\to\infty,\ p_n^r(xq^n\,;s,\ a,\ b\,;\ q)\to V_r(x;\ s\,,\ a\,,\ b\,;q)$ almost everywhere. Hence, by using Fatou's lemma, we get the inequality:

$$\int_E V_r(x;\ s\,,\ a\,,\ b\,;q)\ dm\ \le\ \varliminf_{n\to\infty}\int_E p_n^r(xq^{sn}\,;\ s,\ a,\ b\,;\ q)\ dm \tag{18}$$

3. Conclusion

The results obtained in this paper are seems to be new and quite interesting for the scope of further research. These results give the connection between Measure theory and Special Functions in particular q-analogue.

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