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Expression of a Real Matrix as a Difference of a Matrix and its Transpose Inverse

Expresión de una matriz real como diferencia de una matriz y su transposición inversa

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Abstract. In this paper we derive a representation of an arbitrary real matrix $\mathbf{M}$ as the difference of a real matrix $\mathbf{A}$ and the transpose of its inverse. This expression may prove useful for progressing beyond known results for which the appearance of transpose-inverse terms prove to be obstacles, particularly in control theory and related applications such as computational simulation and analysis of matrix representations of articulated figures.

Keywords. Articulated figure analysis; algebraic Riccati equation; control systems; matrix analysis.

Resumen. En este artculo derivamos una representación de una matriz real arbitraria $\mathbf{M}$ como la diferencia de una matriz real $\mathbf{A}$ y la transposición de su inversa. Esta expresión puede resultar útil para progresar más allá de los resultados conocidos para los cuales la aparición de términos de transposición-inversa resulta ser un obstáculo, particularmente en la teoría de control y aplicaciones relacionadas como la simulación computacional y el análisis de representaciones matriciales de figuras articuladas.

Palabras Claves. Analisis de figura articulada; ecuación algebraica de Riccati; sistemas de control; análisis matricial.

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1. Introduction

A common way to glean information about a given matrix, e.g., to reveal opportunities for manipulating or transforming it, is to express it as the sum or difference of matrices with particular structure or properties. This may be a simple expression of a singular matrix as a sum of nonsingular ones [7] or a given matrix expressed as the sum of a symmetric matrix and a skew-symmetric matrix [3]. *Matrix splitting* is an example of a widely-used technique that relies on the expression of a matrix as the difference of two matrices with special properties, e.g., for solving systems of differential equations [16].

Transpose-inverse terms $(\mathbf{A}^{-1})^{\mathrm{T}}$ $\left(\text{or, equivalently, } (\mathbf{A}^{\mathrm{T}})^{-1}\right)$, commonly abbreviated as $\mathbf{A}^{-\mathrm{T}}$, arise naturally in a variety of control system contexts, e.g., the relative gain array (RGA) [1] and

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formulations of the controllability Gramian [5]. As an example, classical solution methods for the discrete-time algebraic Riccati equation involve the symplectic form [6]:

$$\begin{bmatrix} \mathbf{F} + \mathbf{G}\mathbf{F}^{\text{-T}}\mathbf{H} & -\mathbf{G}\mathbf{F}^{\text{-T}} \\ -\mathbf{F}^{\text{-T}}\mathbf{H} & \mathbf{F}^{\text{-T}} \end{bmatrix} \tag{1}$$

Unfortunately, there are very few matrix factorizations or decompositions involving transpose-inverse terms to aid in the manipulation of equations for theoretical analysis or practical implementation. In this paper we provide an incremental improvement to this state of affairs with a representation of an arbitrary nonsingular real matrix $\mathbf{M}$ as the difference of a real matrix $\mathbf{A}$ and its transpose inverse:

$$\mathbf{M} \;=\; \mathbf{A} - \mathbf{A}^{\text{-T}}. \tag{2}$$

We begin with a derivation of this result in the next section and then develop related results for singular and complex matrices.

2. The Real Nonsingular Case

The main result can be derived from an application of the singular value decomposition (SVD) of the real nonsingular matrix $\mathbf{M}$ as

$$\mathbf{M} \;=\; \mathbf{U}\mathbf{D}\mathbf{V}^{\text{T}} \tag{3}$$

where $\mathbf{U}$ and $\mathbf{V}$ are real orthonormal and $\mathbf{D}$ is a positive diagonal matrix of the singular values of $\mathbf{M}$. It can be observed that defining $\mathbf{A}$ with diagonal $\mathbf{E}$ as

$$\mathbf{A} \;=\; \mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}} \tag{4}$$

satisfies

$$\begin{aligned} \mathbf{M} \;&=\; \mathbf{A} - \mathbf{A}^{\text{-T}} && (5)\\ &=\; \mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}} - (\mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}})^{\text{-T}} && (6)\\ &=\; \mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}} - \left((\mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}})^{\text{-1}}\right)^{\text{T}} && (7)\\ &=\; \mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}} - (\mathbf{V}\mathbf{E}^{\text{-1}}\mathbf{U}^{\text{T}})^{\text{T}} && (8)\\ &=\; \mathbf{U}\mathbf{E}\mathbf{V}^{\text{T}} - \mathbf{U}\mathbf{E}^{\text{-1}}\mathbf{V}^{\text{T}} && (9) \end{aligned}$$

only if each diagonal element $\mathbf{E}_{ii}$ satisfies

$$\mathbf{D}_{ii} \;=\; \mathbf{E}_{ii} - \mathbf{E}_{ii}^{\text{-1}}. \tag{10}$$

This defines a quadratic constraint on each $\mathbf{E}_{ii}$ that can be verified to admit solutions

$$\mathbf{E}_{ii} \;=\; \frac{1}{2}\left(\mathbf{D}_{ii} \pm \sqrt{\mathbf{D}_{ii}^2 + 4}\right) \tag{11}$$

which can be taken as real for all $\mathbf{D}_{ii}$ by positivity of the singular values of $\mathbf{M}$. Furthermore, the nonnegative solution can be taken for each $\mathbf{E}_{ii}$ so that $\mathbf{A}$ is determined completely up to the uniqueness (or lack thereof) provided by the SVD evaluation method.

The case of singular $\mathbf{M}$ can be handled in two ways. As it stands, the previous construction can be restricted to only the nonzero singular values to give

$$\mathbf{M} \;=\; \mathbf{A} - (\mathbf{A}^{\dagger})^{\text{T}} \tag{12}$$

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where $\mathbf{A}$ now has the same rank as $\mathbf{M}$ and its inverse is replaced with a pseudoinverse. Alternatively, the zero singular values of $\mathbf{A}$ above can be replaced with unity[2] so that singular $\mathbf{M}$ is expressed as the difference of nonsingular matrices in the form of Eq. (2). In summary, the expression of Eq. (2) can be obtained for all real square matrices $\mathbf{M}$ whereas that of Eq. (12) extends to real $\mathbf{M}$ of any shape.

3. Variations and Generalizations

The results from the previous section can be verified to generalize directly to the case of complex $\mathbf{M}$ if conjugate-transpose is used in place of the transpose operator

$$\mathbf{M} \;=\; \mathbf{A} - (\mathbf{A}^{-1})^{*}. \tag{13}$$

However, variants in which the matrix difference of Eq. (2) is replaced with a sum, or the transpose operator is maintained for complex $\mathbf{M}$, cannot generally be obtained in a similar form.

The challenge posed by the form

$$\mathbf{M} \;=\; \mathbf{A} + \mathbf{A}^{-\mathrm{T}} \tag{14}$$

is that the resulting analog of Eq. (11) becomes

$$\mathbf{E}_{ii} \;=\; \frac{1}{2}\left(\mathbf{D}_{ii} \pm \sqrt{\mathbf{D}_{ii}^{2} - 4}\right) \tag{15}$$

which admits real solutions only for $\mathbf{D}_{ii} \geq 2$. In other words, the form of Eq. (14) can only be obtained with real $\mathbf{A}$ if the smallest singular value of $\mathbf{M}$ satisfies $\sigma_{\min} \geq 2$. On the other hand, from this we can conclude that the form of Eq. (14) can always be obtained for $\frac{2}{\sigma_{\min}}\mathbf{M}$, i.e., when real nonsingular $\mathbf{M}$ is scaled to ensure its smallest nonzero singular value is 2. This yields the slightly less pleasing form

$$\mathbf{M} \;=\; c\,(\mathbf{A} + \mathbf{A}^{-\mathrm{T}}) \tag{16}$$

which can be obtained for all real nonsingular $\mathbf{A}$ by letting $c = 2/\sigma_{min}$.

For completeness we note that Eq. (11) & Eq. (15) can be applied with matrix arguments in place of diagonal elements to obtain, respectively, the non-transpose forms:

$$\mathbf{M} \;=\; \mathbf{A} - \mathbf{A}^{-1} \tag{17}$$

and

$$\mathbf{M} \;=\; c\,(\mathbf{A} + \mathbf{A}^{-1}) \tag{18}$$

which are not relevant to the focus of this paper, e.g., because they are not applicable to rectangular matrices, but may be of independent interest.

4. Application to Articulated Figure Analysis

For purposes of kinematic simulation and analysis of articulated figures, e.g., for motion prediction and/or animation, there are multiple mathematical representations. Most commonly, a human figure is represented with segments and joints (see Fig. 1) with the mobility constraints expressed using Euler angles, quaternions (or double quaternions), and/or exponential maps [15, 11, 2].

[2] This works because the unit singular values will cancel in the difference with their corresponding ones in the transpose inverse, and in fact 1 is the positive solution to Eq. (11) for $\mathbf{D}_{ii} = 0$.

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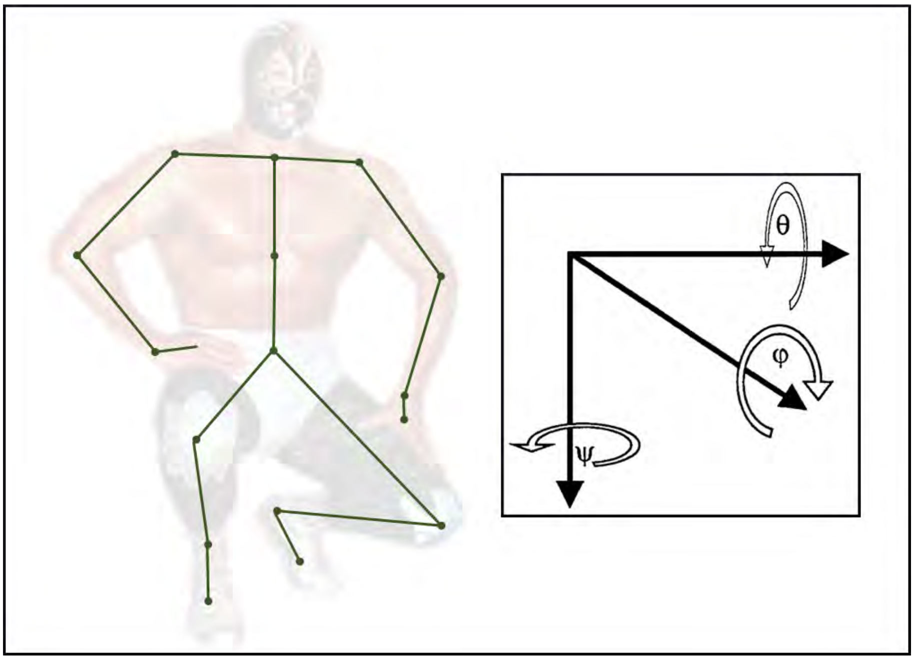

Figure 1: Segment and joint representation of an articulated human figure.

More generally, the structure of the figure may be represented simply as a matrix. For example, compositions of rigid rotations involving a heirarchy of coordinate axes can be represented in the form of a real orthonormal rotation matrix, $\mathbf{R}$, such that the transformation of a given matrix $\mathbf{M}$ representing an articulated figure can be expressed as $\mathbf{RMR}^{\mathrm{T}}$. The price paid for the generality of representing figures and shapes using matrices is the challenge of how to manipulate and analyze such expressions, e.g., for operations such as graph matching or simply to gain enhanced intuitive insights. As indicated in [9]: "it is not yet clear how to choose and characterize the group of transformations under which such shapes should be studied."

A key property that clearly must be maintained is consistency with respect to real orthonormal transformations. It can easily be verified that this property is maintained by the transformation $f(\mathbf{M}) \to \mathbf{A}$, $\mathbf{M} = \mathbf{A} - \mathbf{A}^{\text{-T}}$, by virtue of its derivation via the SVD, or explicitly as:

$$\begin{aligned} \mathbf{RMR}^{\mathrm{T}} &= \mathbf{R}(\mathbf{A} - \mathbf{A}^{\text{-T}})\mathbf{R}^{\mathrm{T}} && (19) \\ &= \mathbf{RAR}^{\mathrm{T}} - \mathbf{RA}^{\text{-T}}\mathbf{R}^{\mathrm{T}} && (20) \\ &= (\mathbf{RAR}^{\mathrm{T}}) - (\mathbf{RAR}^{\mathrm{T}})^{\text{-T}} && (21) \end{aligned}$$

where the fact that $\mathbf{R}^{\text{-T}} = \mathbf{R}$ for real orthonormal $\mathbf{R}$ has been exploited. This therefore demonstrates the desired consistency property:

$$f(\mathbf{RMR}^{\mathrm{T}}) \;=\; \mathbf{R} \cdot f(\mathbf{M}) \cdot \mathbf{R}^{\mathrm{T}}. \qquad (22)$$

In the previous section we discussed the generalization from transpose to conjugate-transpose, which can support consistency with respect to unitary transformations, and it is natural to consider a further generalization to the nonassociative octonions. This would permit forces to be more flexibly incorporated [17], and it would permit temporal sequences of non-compositional operators to be expressed uniquely based on a specified associativity rule [8]. More specifically, a temporal sequence of non-associative operators α_i applied at times t_i could be derived as a

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solution to a given problem and expressed in directional time-assymetric form as

$$\begin{array}{lcl} t_0 & \rightarrow & \alpha_0 \\ t_1 & \rightarrow & \alpha_1(\alpha_0) \\ t_2 & \rightarrow & \alpha_2(\alpha_1(\alpha_0)) \\ & \vdots & \end{array} \tag{23}$$

Unfortunately, the proposed matrix decomposition does not appear to practically accommodate unitary transformations over the octonions because there presently does not exist an efficiently computable octonion analog of the SVD[3].

5. Discussion

The main result of this paper is that every real matrix $\mathbf{M}$ can be represented as the difference of two real matrices in the form

$$\mathbf{M} \;=\; \mathbf{A} - \mathbf{A}^{-\mathrm{T}}. \tag{24}$$

A measure of the potential utility of a given factorization or decomposition is the extent to which it can be used to obtain nontrivial derivative results. Eq. (24) admits a variety of trivial ones, e.g., multiplication of $\mathbf{M}$ by $\mathbf{A}^{-1}$ or $\mathbf{A}^{\mathrm{T}}$ gives a symmetric difference involving a positive semidefinite matrix and the identity matrix. However, additional structural properties can be derived based on known results involving transpose inverses, e.g., the relative gain array (RGA) mentioned in the introduction. The RGA of a square nonsingular matrix $\mathbf{G}$ is defined [1] as

$$\mathcal{R}GA(\mathbf{G}) \;\doteq\; \mathbf{G} \circ (\mathbf{G}^{-1})^{\mathrm{T}} \tag{25}$$

where $\circ$ represents the elementwise Hadamard matrix product. Because of its practical importance in control applications the RGA has been shown [4] to have a variety of interesting mathematical properties, and those properties therefore carry over to the Hadamard product of the matrix terms of Eq. (24). For example, it is known that $\mathcal{R}GA(\mathbf{G})$ is invariant with respect to diagonal scalings of $\mathbf{G}$ and that its rows and columns have unit sum, i.e., $\mathcal{R}GA(\mathbf{G})$ is generalized doubly stochastic. This implies that by letting $\mathbf{B} = \mathbf{A}^{-\mathrm{T}}$ in Eq. (24) one can obtain an expression with significantly less explicit structural information

$$\mathbf{M} \;=\; \mathbf{A} - \mathbf{B}, \quad \mathbf{A} \circ \mathbf{B} \in \textit{Generalized Doubly Stochastic} \tag{26}$$

which would still be of interest based solely on the property that $\mathbf{A} \circ \mathbf{B}$ is generalized doubly stochastic[4]. In summary, the main result of this paper is a specialized form of matrix splitting that offers a diverse set of interesting mathematical properties that derive from and thus may prove useful to control theory and related applications.

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[3] However, the recent approach of [10] is of potential relevance in this regard.
[4] It has recently been shown that a new generalized inverse [12] can be used to preserve some RGA properties in the case of singular $\mathbf{G}$ [13, 14], thus conferring related properties to the expression of Eq. (12).

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