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PHYSICS

Lanczos spintensor in Gödel spacetime via factorization of the metric tensor

Espintensor de Lanczos en el espacio-tiempo de Gödel a través de la factorización del tensor métrico

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Abstract. We show a factorization of the metric tensor in Gödel cosmological model which gives a generator for the Lanczos spintensor of the conformal tensor.

Keywords. Gödel geometry; Lanczos potential; Weyl tensor.

Resumen. Mostramos una factorización del tensor métrico en el modelo cosmológico de Gödel, lo cual da un generador para el espíntensor de Lanczos del tensor conformal.

Palabras Clave. Geometría de Gödel; Potencial de Lanczos; Tensor de Weyl.

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The Lanczos potential K_{abc} has the properties [1–3]:

$$K_{abc} = -K_{bac}, \qquad K_{abc} + K_{bca} + K_{cab} = 0, \qquad K_a{}^b{}_b = 0, \qquad K^{abc}{}_{;c} = 0, \quad (1)$$

and it permits to construct the conformal tensor in according with the fundamental relation [1]:

$$C_{abcd} = K_{abc;d} - K_{abd;c} + K_{cda;b} - K_{cdb;a} + K_{ad}\, g_{bc} - K_{ac}\, g_{bd} + K_{bc}\, g_{ad} - K_{bd}\, g_{ac}, \qquad (2)$$

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where:

$$K^{ab} = K^{ba} = K^{acb}{}_{;c} \ . \tag{3}$$

For the Gödel's geometry [4–7]:

$$ds^2 = \left(dx^0\right)^2 + 2\ e^{x^3} dx^0\ dx^1 + \frac{1}{2}\ e^{2x^3}\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2, \tag{4}$$

we can write the following factorization of the metric tensor:

$$\left(g^{ab}\right) = \begin{pmatrix} -1 & 2e^{-x^3} & 0 & 0 \\ 2e^{-x^3} & -2e^{-2x^3} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left(P^{ac}\ Q_c{}^b\right), \tag{5}$$

such that:

$$\left(P^{ac}\right) = \left(\frac{1}{9}\ E^{ac}\right) = \frac{1}{9} Diag\left(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \quad \left(Q_c{}^b\right) = 9\begin{pmatrix} \frac{4}{3} & -\frac{8}{3} & 0 & 0 \\ 8e^{-x^3} & -8e^{-2x^3} & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}, \tag{6}$$

where E^{ac} is the Ricci tensor without trace. It is surprising that the factor P^{ac} generates the Lanczos potential for this Gödel cosmological model:

$$K_{abc} = P_{cb;a} - P_{ca;b} = \frac{1}{9}\left(R_{cb;a} - R_{ca;b}\right), \ \ \left(R_{ab}\right) = \begin{pmatrix} -1 & -e^{x^3} & 0 & 0 \\ -e^{x^3} & -e^{2x^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \ R = R^a{}_a = -1, \tag{7}$$

thus (2), (3) and (7) imply the following components different to zero:

$$\begin{aligned} &K_{031} = K_{103} = \frac{1}{2}\ K_{130} = \frac{1}{18}\ e^{x^3}, \qquad && K_{131} = \frac{1}{6}\ e^{2x^3}, \\ &-\frac{1}{2}K^0{}_0 = K^1{}_1 = K^3{}_3 = \frac{1}{6}, && K^0{}_1 = -\frac{1}{2}\ e^{x^3}. \end{aligned} \tag{8}$$

$$C_{0303} = -C_{2323} = -\frac{1}{2}C_{0202} = \frac{1}{6}, \ \ C_{0313} = \frac{1}{2}C_{1220} = \frac{1}{6}e^{x^3}, \ \ C_{0101} = \frac{1}{4}C_{1313} = -\frac{1}{5}C_{1212} = \frac{1}{12}e^{2x^3}.$$

Besides, it is possible to show the relation:

$$K^{abc} = -\frac{2}{9}\ C^{abcr}{}_{;r} \ , \tag{9}$$

which may be considered as the inversion of (2), that is, the Weyl tensor generates the Lanczos potential in Gödel spacetime. It is possible to deduce the wave equation [8] for some quantities of interest, for example:

$$\Box K^{abc} \equiv K^{abc;r}{}_{;r} = -3\ K^{abc}, \qquad \Box R_{ab} = 2\ \Box K_{ab} = -6\ K_{ab}, \tag{10}$$

and several identities involving the Einstein, Ricci and Riemann tensors:

$$K_{ab;c} + K_{bc;a} + K_{ca;b} = 0, \qquad K^{ab}{}_{;b} = 0, \qquad R_{abcr}\ K^{abc} = 0,$$
$$R^r{}_{[a}K_{b]cr} - R^r{}_c K_{abr} - R^r{}_q K_{r[a}{}^q g_{b]c} = K_{abc}, \qquad K_{ab} = \frac{1}{6}\left(R_{ab} - 2R_{acrb}\ G^{cr}\right), \tag{11}$$

and K_{ab} turns out to be a potential for the Lanczos tensor:

$$K_{abc} = -\frac{2}{9}\ \left(K_{ca;b} - K_{cb;a}\right), \tag{12}$$

thus (12) is the inversion of (3). We indicate the following non-zero components:

$$\Gamma_{013} = \Gamma_{103} = -\Gamma_{301} = \frac{1}{2}\, e^{x^3},\ \Gamma_{113} = -\Gamma_{311} = \frac{1}{2}\, e^{2x^3},\ \Gamma^3{}_{01} = \Gamma^0{}_{13} = \frac{1}{2}\, e^{x^3},\ \Gamma^0{}_{03} = 1,\ \Gamma^1{}_{03} = -e^{-x^3},$$

$$\Gamma^3{}_{11} = \frac{1}{2}e^{2x^3}, \qquad R_{0303} = \frac{1}{2}\,, \qquad R_{0313} = \frac{1}{2}\ e^{x^3}, \qquad R_{1313} = 3\ R_{0101} = \frac{3}{4}e^{2x^3}, \tag{13}$$

$$(G^a{}_c) = \begin{pmatrix} -\frac{1}{2} & -e^{x^3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

An adequate linear combination of R_{ab} and K_{ac} gives an interesting generator:

$$K_{abc} = -\frac{\sqrt{2}}{18}\ \left(B_{ca;b} - B_{cb;a}\right), \qquad B^{ac}{}_{;c} = 0,$$
$$B_{ac} = \frac{1}{\sqrt{2}}\left(6K_{ac} - R_{ac}\right), \qquad \Box B_{ac} = -6\sqrt{2}\ K_{ac}, \tag{14}$$

and it is remarkable the fulfillment of the Gauss equation [7, 9–11]:

$$R_{abcr} = B_{ac}\ B_{br} - B_{ar}\ B_{bc}, \qquad B_{ac} = -\sqrt{2}\ R_{abrc}\ G^{br}, \qquad \overset{*}{\Box} R^{*abcr}\ R_{abcr} = 0, \tag{15}$$

hence B_{ac} generates the Lanczos potential via the differential relation (14) and also the curvature tensor through the algebraic expression (15).

The idea of factoring the metric tensor, as in (5), to obtain a generator of the Lanczos potential, also works in Kerr spacetime [12].

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