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Mathematics - Short Article

On certain combinatorial identities

Sobre ciertas identidades combinatorias

A. Hernández-Galeana[+], E. Santiago-Cortés[#&] and J. López-Bonilla[*] [1]

[+]Departamento de Física, ESFM, Instituto Politécnico Nacional, México

[#]Facultad de Ingeniería, Corporación Universitaria Autónoma del Cauca, Colombia

[&]Departamento de Ciencias, Escuela de Ingeniería y Ciencias, Tecnologico de Monterrey - ITESM, Campus Santa Fe, México

[*]ESIME-Zacatenco, Instituto Politécnico Nacional, México

Abstract. We study combinatorial identities associated with the normalized binomial mid-coefficients and the self-conjugate permutations.

Keywords. Hypergeometric function, Petkovsek-Wilf-Zeilberger's method, Binomial coefficient, Self-conjugate permutations.

Resumen. Estudiamos identidades combinatorias asociadas con los mid-coeficientes binomiales normalizados y las permutaciones auto-conjugadas.

Palabras Clave. Función hipergeométrica, Método de Petkovsek-Wilf- Zeilberger, Coeficiente binomial, Permutación auto-conjugada.

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The following expression is considered in [1]:

$$A \equiv \sum_{k=0}^{n} (-1)^k \binom{2\,k}{k} \binom{2\,(n-k)}{n-k}\,, \tag{1}$$

whose value is in the relation (7.4) of [2] and in [3]:

$$A = \frac{1+(-1)^n}{2}\, 2^n \binom{n}{n/2} = \begin{cases} 0 & , n \text{ odd} \\ 2^n \dbinom{n}{n/2} & , n \text{ even} \end{cases}\,, \tag{2}$$

[1] *Corresponding author: jlopezb@ipn.mx*

We can indicate the identity [2-4]:

$$\sum_{k=0}^{n} \binom{2\,k}{k} \binom{2\,(n-k)}{n-k} = 4^n \,, \tag{3}$$

and the formula (6.75) of [2]:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{2\,j}{j} (\frac{z}{4})^j = \frac{1}{2^{2\,n}} \sum_{k=0}^{n} (-1)^k \binom{2\,k}{k} \binom{2\,(n-k)}{n-k} (z-1)^k \,, \tag{4}$$

where we can apply (2) and $z = 2$ to obtain the property:

$$\sum_{j=0}^{n} (-\frac{1}{2})^j \binom{2\,j}{j} \binom{n}{j} = \frac{1+(-1)^n}{2} \frac{1}{2^n} \binom{n}{n/2} = \begin{cases} 0 & , n \text{ odd} \\ 2^{-n} \binom{n}{n/2} & , n \text{ even} \end{cases} \,, \tag{5}$$

involving the normalized binomial mid-coefficients:

$$\mu_m = \sum_{j=0}^{2\,m} \frac{(-1)^j}{2^j} \binom{2\,j}{j} \binom{2\,m}{j} = \frac{1}{4^m} \binom{2\,m}{m} = \frac{1}{m!} \left(\frac{1}{2}\right)_m = \frac{(2\,m-1)!!}{(2\,m)!!} = \frac{1}{m!\,\sqrt{\pi}}\, \Gamma(m+\frac{1}{2}) \,, \tag{6}$$

which verify interesting identities, for example:

$$\sum_{k=0}^{\infty} \frac{\mu_k}{4^k\,(2\,k+1)^3} = \frac{7\,\pi^3}{216} \,, \qquad [5] \tag{7}$$

$$\sum_{k=2}^{\infty} \frac{1}{2^k\,\mu_k} = \frac{\pi}{2} \,, \qquad \sum_{k=1}^{\infty} \frac{\mu_k}{k+1} = 1 \,, \qquad [6] \tag{8}$$

$$\frac{1}{\mu_m^2} = \pi\, m\; {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{2}; m; 1\right) \,, \qquad [7] \tag{9}$$

$$\sum_{k=1}^{\infty} \frac{\mu_k\, H_k}{k} = \frac{\pi^2}{3} \,, \qquad [8] \tag{10}$$

involving harmonic numbers [9] and the Gauss hypergeometric function [10].

If U_n is the number of self-conjugate permutations of $\{1, 2, \ldots, n\}$, then [11]:

$$U_n = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!\,(n-2\,k)!\,2^k} \,, \tag{11}$$

where we can apply the Petkovsek-Wilf-Zeilberger's method explained in [10, 12-20], to deduce the relations:

$$U_{2m} = {}_2F_0\left(-m, \frac{1}{2}-m; -; 2\right) = (2m-1)!!\, {}_1F_1\left(-m, \frac{1}{2}; -\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^m H_{2m}\left(-\frac{i}{\sqrt{2}}\right),$$

$$U_{2m+1} = {}_2F_0\left(-m-\frac{1}{2}; -m; -; 2\right) = (2m+1)!!\, {}_1F_1\left(-m, \frac{3}{2}; -\frac{1}{2}\right) = \frac{i}{\sqrt{2}}\left(-\frac{1}{2}\right)^m H_{2m+1}\left(-\frac{i}{\sqrt{2}}\right), \tag{12}$$

involving Hermite polynomials [14].

In [21] is the following expression:

$$I_x(k, n-k+1) = \sum_{j=k}^{n} \binom{n}{j} x^j (1-x)^{n-j}, \qquad I_x(\alpha,\beta) = \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)},$$

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt, \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{13}$$

where we can employ the value $x = \frac{1}{2}$ to obtain:

$$\sum_{j=k}^{n} \binom{n}{j} = 2^n k \binom{n}{k} \int_0^{1/2} t^{k-1}(1-t)^{n-k}dt = 2^{n-k} k \binom{n}{k} \sum_{r=0}^{n-k} \binom{n-k}{r} \frac{(-\frac{1}{2})^r}{r+k}, \tag{14}$$

On the other hand, from (6.36) of [2]:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{x^{r+k}}{r+k} = \sum_{j=1}^{r} (-1)^{r-j} \binom{r-1}{r-j} \frac{(x+1)^{j+n}-1}{j+n}, \quad r \geq 1, \tag{15}$$

hence:

$$\sum_{r=0}^{n-k} \binom{n-k}{r} \frac{(-\frac{1}{2})^r}{r+k} = 2^k \sum_{j=1}^{k} (-1)^j \binom{k-1}{k-j} \frac{(\frac{1}{2})^{j+n-k}-1}{j+n-k}, \tag{16}$$

which can be applied into (14).

We deduced, via arithmetical experimentation, the property [22-24]:

$$\sum_{r=k}^{n} r\, S_n^{(r)} S_r^{[k]} = (-1)^{n+k+1}(n-k-1)! \binom{n}{k-1}, \qquad 1 \leq 1+k \leq n, \tag{17}$$

which for $k = 1, 2, 3$ and the following values for $r \geq 1$ [25, 26]:

$$S_r^{[1]} = 1, \qquad S_r^{[2]} = 2^{r-1} - 1, \qquad S_r^{[3]} = \frac{1}{2}(3^{r-1} + 1 - 2^r), \tag{18}$$

imply the relations:

$$\sum_{r=1}^{n} r\, S_n^{(r)} = \begin{cases} 1 & , n=1\,, \\ (-1)^n\,(n-2)! & , n\geq 2\,, \end{cases} \quad , \sum_{r=2}^{n} r\, 2^r\, S_n^{(r)} = \begin{cases} 8 & , n=2\,, \\ 2(-1)^n\,(n-3)!\, n\,(n-3) & , n\geq 3\,, \end{cases}$$

$$\sum_{r=3}^{n} r\, 3^r\, S_n^{(r)} = \begin{cases} 81 & , n=3\,, \\ 3\,(-1)^n\,(n-4)!\,[n(n-1)+(n-3)(n(n-4)-6(n-1)(n-2)H_{(n-1)})] & , n\geq 4\,, \end{cases} \tag{19}$$

Thus, it is evident that (17) gives several identities for various values of k, involving the Stirling and harmonic numbers. Besides, the application of $\sum_{k=0}^{n}$ to (17) generates the interesting property:

$$\frac{1}{n}\sum_{k=1}^{n} k\, S_n^{(k)}\, B(k) = 1 + (n-1)! \sum_{j=0}^{n-2} \frac{(-1)^{n+j}}{j!\,(n-j)(n-j-1)}\,, \qquad n\geq 1\,, \tag{20}$$

involving the Bell numbers [26-28].

We have the following Todorov's expression for the generalized Bernoulli numbers in terms of the Stirling of the second kind [21, 29]:

$$B_k^{(n)} = \sum_{j=0}^{k} (-1)^j\, \frac{\binom{n+k}{k-j}\binom{n+j-1}{j}}{\binom{k+j}{j}}\, S_{k+j}^{[j]}\,, \tag{21}$$

where we can employ the Carlitz's identity [30, 31]:

$$B_k^{(n)} = \frac{1}{\binom{k-n}{k}}\, S_{k-n}^{[-n]} = \frac{1}{\binom{n-1}{k}}\, S_n^{(n-k)}\,, \tag{22}$$

to obtain the Schläfli's formula [26, 32]:

$$S_n^{(n-k)} = (-1)^k \sum_{j=0}^{k} \binom{k+n}{k-j}\binom{k-n}{k-j}\, S_{k+j}^{[j]}\,. \tag{23}$$

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