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Topology

Polynomial Maps of Spheres

Mapas de polinomios de esferas

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Abstract. The real multiplication map $\varnothing_{m,m} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{2m-1}$ induces a symetric immersion $\widetilde{\varnothing}_m : S^{m-1} \to \mathbb{R}P^{m-1} \to \mathbb{R}^{2m-2}$ which by a theorem of E.H.Brown has mod two Whitney invariant 1 if and only if $m = 2^p$ for some $p \geq 1$. As an explanation of this fact we provide an explicit regular homotopy from the immersion $\widetilde{\varnothing}_m$ to another map essentially given by a polynomial self map of S^{m-1} whose degree equals the Whitney invariant of $\widetilde{\varnothing}_m$ mod 2. Another choice of a polynomial self-map of S^{m-1} yields an immersion in the regular homotopy class of $\widetilde{\varnothing}_m$ whose Whitney invariant is visible from its double point set.

Keywords. Topology; Polynomial Maps; Spheres.

Resumen. El mapeo de multiplicación polinomial $\varnothing_{m,m} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{2m-1}$ 1induce una inmersión simétrica $\widetilde{\varnothing}_m : S^{m-1} \to \mathbb{R}P^{m-1} \to \mathbb{R}^{2m-2}$ la cual, por un teorema de E.H. Brown tiene invariante de Whitney 1 si y solo si $m = 2^p$ para algún $p \geq 1$. Para explicar este hecho, nosotros exhibimos una homotopía regular entre la inmersión $\widetilde{\varnothing}_m$ y otro mapeo dado esencialmente por un auto mapeo polinomial de S^{m-1} cuyo grado es el invariante de Whitney de $\widetilde{\varnothing}_m$ mod 2. AnotherOtra selección de auto mapeo polinomial de S^{m-1} conduce a una clase de homotopía regular de $\widetilde{\varnothing}_m$ cuyo invariante de Whitney es visible desde su conjunto de puntos dobles.

Palabras Clave. Topología; Mapas de polinomios; Esferas.

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1. Introduction

Let $\varnothing_{m,k} : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^{m+k-1}$ be the non-singular bilinear map given by polynomial multiplication: for $x = (x_0, \ldots, x_{m-1}), y = (y_0, \ldots, y_{k-1})$, $\varnothing_{m,k}(x, y) = (z, \ldots, z_{m+k-2})$, where $z = (z_0, \ldots, z_{m+k-2})$ and $z_h = \sum_{i+j=h} x_i y_j$. $\varnothing_{m,m}$ determines a symmetric immersion $\widetilde{\varnothing}_m : S^{m-1} \to \mathbb{R}P^{m-1} \to \mathbb{R}^{2m-2}$ by restriction to the diagonal sphere and normalization of its image. By a theorem of E.H. Brown [2] the mod 2 Whitney invariant of any symmetric immersion (and hence of $\widetilde{\varnothing}_m$) is nonzero if and only if $m = 2^p$ for some $p \geq 1$.

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About the adjoint $f_{m,k}$ of $\varnothing_{m,k}$, K.Y.Lam [4] proved the following result. Recall [6] that the group $\pi_{m-1}(V_{m+k-1,k})$ is cyclic; let e_{m-1}denote a generator.

Theorem 1. (K.Y.Lam) Let $m \geq 1$ and $k \leq 1$

a) For $m = 2p, \quad k = 2h - i, \quad i = 0, 1$ $[f_{m,k}] = \binom{p+h-1}{p} e_{m-1} \in \mathbb{Z}_2$ or Z *as* $k \neq 1$ or k=1

b) For $m = 2p + 1, \quad k = 2h - 1$, $[f_{m,k}] = \binom{p+h-1}{p} e_{m-1} \in \mathbb{Z}$

c) For $m = 2p + 1, \quad k = 2h$, $[f_{m,k}] = 0 \in \mathbb{Z}$.

This theorem is an easy consequence of the next result and a computation of the degree of a map $P_{m,k}$ which we now define:

For $m \geq 1, \quad k \geq 0$ let $P_{m,k} : \ \mathbb{R}^m \to \mathbb{R}^m$ be the polynomial map defined by the matrix equation $P_{m,k}(x) = x\{\ M(x)\}^{\ k}$ where x is the position row vector $x = (x_0, \ldots, x_{m-1})$ and $M(x)$ is the $m \times m$ matrix whose first $m-1$ rows are (in order) the $2^{\text{nd}}, \ldots, m^{\text{th}}$ standard vectors of $\mathbb{R}^m$ and whose last row is $-x$. We can extend the definition of $P_{m,k}$ by setting $P_{m,k} = (-1)$ times the $(m+k-1)^{\text{st}}$ standard basis vector of $\mathbb{R}^m$ when $-(m-1) \leq k \leq -1$. It follows that $M(x)$ has rows $-P_{m,-(m-1)}, \ldots, -P_{m,-1},\ P_{m,0}(=-x)$

Theorem 2. For $m \geq 1,\ k \geq 1$ there is a homotopy commutative diagram:

$$\begin{tikzcd}
 & V_{m,1} \arrow[d, "i"] \\
S^{m-1} \arrow[ur, "P_{m,k-1}"] \arrow[r, "f_{m,k}"'] & V_{m+k-1,k}
\end{tikzcd}$$

The proof of this theorem consists of an explicit homotopy given in 3. A calculation of the degree of , $P_{m,k}$ is given in 2.

The above discussion has a complex analog (e.g., a complex version of theorem 2) with $\mathbb{R}^m$ replacing $\mathbb{R}^m$. While we do not use this fact, we do calculate the degree of the analogous complex polynomials $Q_{m,k}$.

In 4. we relate theorems 1 and 2 to a question of L.Smith which ask for an explanation of Brown's theorem for the immersions $\widetilde{\varnothing}_m$. Also, in 4. as a further application, we offer a minor simplification of Smith´s proof [5] of Brown's theorem.

2. The polynomials

The homotopies we construct in 3. lead to the polynomial maps $P_{m,k}$ defined in 1. The latter satisfy the recurrence relation:

$$\text{(2.1)} \qquad \mathrm{P}_{m,k}(x) = P_{m,k-1}(x)\ \mathrm{M}(x), \qquad k \geq -(m-1)$$

Following a suggestion of T.Iwaniec, we consider the polynomial (in $\mathbb{R}[x_0, x_1, \ldots, x_{m-1}][\lambda]$)

$$\text{(2.2)} \qquad \mathrm{R}_{m,k}(\lambda) = \left(\sum_{0 \leq i \leq m-1} [P_{m,k}(x)]_i \lambda^i \right) + \lambda^{m+k}$$

Where $[P_{m,k}(x)]_i$ denotes the i^{th} component of $P_{m,k}(x)$. Note that $R_{m,0}(\lambda) = \left(\sum_{0\leq i\leq m-1} x_i\lambda^i\right) + \lambda^{m+k}$ and in fact that the transpose $M(x)^T$ of $M(x)$ is the companion matrix of $R_{m,0}(\lambda)$. We can restate (2.1) as

$$\mathrm{R}_{m,k}(\lambda) = \ R_{m,k-1}(\lambda) - [P_{m,k-1}(x)]_{m-1}\ \mathrm{R}_{m,0}(\lambda)$$

By an easy induction $\mathrm{R}_{m,k}(\lambda) = \ \mathrm{S}_{m,k}(\lambda)\ \mathrm{R}_{m,0}(\lambda)$ where

$$S_{m,k}(\lambda) = \lambda^k - \sum_{0\leq i\leq k-1} [P_{m,i}(x)]_{m-1}\lambda^{k-i-1}$$

Lemma (2.1). The map $s_m: \ \mathbb{R}^m \to \mathbb{R}^m$ which assigns to $(\xi_0, \xi_1, \ldots, \xi_{m-1})$ the m-tuple whose i^{th} component is the i^{th} Elementary symmetric function $,\sigma_1(\xi_0, \xi_1, \ldots, \xi_{m-1})$ is a proper function.

The usual proof of the fundamental theorem of algebra contains an estimate which provides an elementary proof of this lemma.

T.Iwaniec has proved the following via an estimate for $\|P_{m,k}\|$.

Corollary (2.2). For $k \geq 0$, $\mathrm{P}_{m,k}: \ \mathbb{R}^m \to \mathbb{R}^m$ is a proper map.

Proof: Let $x = (x_0, \ldots, x_{m-1}) \to \infty$. Because $\mathrm{R}_{m,0}(\lambda)$ has the x_i as coefficients, then some of its roots and hence some root of $\mathrm{R}_{m,k}(\lambda)$ tends to ∞. By lemma (2.1) at least one of the coeficcients of $\mathrm{R}_{m,k}(\lambda)$ tends to ∞, but these are the components of $\mathrm{P}_{m,k}$. So $\mathrm{P}_{m,k}$ is proper.

It is immediate from the definitions that the matrix $M(x)^k$ has rows $-P_{m,k-m}(x), \ldots,\ -P_{m,k-1}(x)$ and so $\mathrm{P}_{m,k}(x) = -x_0\ \mathrm{P}_{m,k-m}(x) - \ldots - x_{m-1}\ \mathrm{P}_{m,k-1}(x)$

Computing the differential directly then obtain

$$\mathrm{DP}_{m,k}(x) = \ \left\{M(x)^k\ \right\}^T - x_0\ \ \mathrm{DP}_{m,k-m}(x) - \ldots - x_{m-1}\ \ \mathrm{DP}_{m,k-1}(x)$$

Hence, we see inductively that $\mathrm{DP}_{m,k}(x)$ is a polynomial in $M(x)^T$ an in fact

$$\mathrm{DP}_{m,k}(x) = {}_{S}{}^{m,k}\left(M(x)^T\right)$$

Since $M(x)^T$ is the companion matrix to $\mathrm{R}_{m,0}(\lambda)$, then $\mathrm{R}_{m,0}(\lambda)$ is the characteristic polynomial of $M(x)^T$. It follows that the determinant

$$(*) \qquad \det\ \mathrm{DP}_{m,k}(x) = \prod_i \mathrm{R}_{m,0}(\beta_i) = \prod_{i,j}(\beta_i - \alpha_j)$$

Where β_i, $i = 1, \ldots, k$ runs over the roots of $\mathrm{S}_{m,k}(\lambda)$ and α_j, $j = 1, \ldots, m$ runs over the roots of $\mathrm{R}_{m,o}(\lambda)$.

Proposition (2.3).

i) The degree of the complex map $Q_{m,k}: \ \mathbb{C}^m \to \mathbb{C}^m$ is the binomial coefficient $\binom{m+h}{k}$

ii) The degree of the real map $P_{m,k}: \ \mathbb{R}^m \to \mathbb{R}^m$ is the binomial coefficient $\binom{p+h}{p}$ if $m = 2p$ and $k = 2h + i$, $i = 0, 1;$ *or* $m = 2p + 1$ *and* $k = 2h$. If $m = 2p + 1$ *and* $k = 2h + 1$ then the degree of $P_{m,k}$ is 0.

Proof:

i) It follows from $(*)$ that if for a given z, $R_{m,k}(\lambda)$ does not have multiple roots, then $P_{m,k}(z)$ is a regular value. Since there are $\binom{m+h}{k}$ factorizations of $R_{m,k}(\lambda)$ as a product of two polynomials, one of degree m and other of degree k, there are $\binom{m+h}{k}$ possibilities for $R_{m,0}(\lambda)$, hence the same number of preimages z' of the regular value $P_{m,k}(z)$.

ii) As in the complex case, we need to determine the number of factorizations of $R_{m,k}(\lambda)$ but now with real factors (as well as the sign of the differential at each preimage of a given regular value).

We consider only the case $m = 2p$, $k = 2h$, as all other cases are similar. Let $R_{m,k}(\lambda) = (\lambda^{2p+2h+2} - 1)/(\ \lambda^2 - 1)$ or in fact any polynomial whit distinct and nonreal roots. A choice of $R_{m,0}(\lambda)$ is now a choice of p of the $p+h$ roots with positive imaginary part (paired with their conjugates). The determinant $\det \mathrm{DP}_{m,k}$ will be then positive so the degree is $\binom{p+h}{p}$.

3. The homotopies

Proof of theorem 2. We provide a sequence of homotopies which together connect the map **Proof**: Let $f_{m,k}$ to the map

$$i\ \circ P_{m,k-1} : S^{m-1} \to V_{m,1} \to V_{m+k-1,k}$$

Recall $f_{m,k}$ is the map

$$f_{m,k}(x) = \begin{pmatrix} x_0 & x_1 & \dots & x_{m-1} & t & & & \\ & x_0 & x_1 & \dots & x_{m-1} & t & & \\ & & \dots & & & & t & \\ & & & x_0 & x_1 & \dots & x_{m-1} & \end{pmatrix}$$

First form the matrix with $(j,\ m+j) - entry$ equal to t for all $1 \leq j \leq k-1$, and with all other entries equal to the corresponding entries of $f_{m,k}$. It lies in $V_{m+k-1,k}$ because if i is the least index such that $x_i \neq 0$, then the matrix admits an upper triangular $k \times k$ submatrix with all diagonal entries equal to x_i. When $t = 1$ let B_0 denote this matrix.

The second step is a sequence of elementary column operations using the $1s$ introduced in the first step (when $t = 1$) to create zeros to the left of these $1s$. This is done one row at a time, starting from the $(k-1)^{\text{st}}$ row and working up to the 1^{st} row, until we arrive to a matrix of the form

$$\begin{pmatrix} 0 & I \\ C & D \end{pmatrix}$$

Where 0 is the zero $(k-1) \times m$ matrix, I the $(k-1) \times (k-1)$ identity matrix, C a $1 \times m$ matrix whose entries are polynomials in the variables $x_0,\ x_1, \dots, x_{m-1}$ and D a $1 \times (k-1)$ matrix of no significance. These elementary column operations are realizable by post multiplication by matrices each homotopic to the identity.

We claim that a C is in fact $P_{m,k-1}(x)$. Let $B_1,\ \dots, B_{k-1}$ be the sequence of matrices obtained from B_0 by clearing out the entries to the left of the $1s$ as described above. Then B_1 is the result of clearing out row $k-1$ in B_0 to the left of its 1, B_2 the result of clearing out row $k-2$ in B_1 to the left of its 1,…, B_{k-1} the result of clearing out row 1 in B_{k-2} to the left of its 1.

The bottom row of B_0 is $O_{k-1}P_{m,0}$ where O_{k-1} denotes $k-1$ $0s$, In using column $m-k+1$ to clear out row $k-1$ by column operations, rows $1, \dots, k-2$ are left unchanged, row $k-1$ has only $0s$ except for the 1 in column $m-k+1$, and row k is altered by the addition of $-x_{m-1}\,(x_0, x_1, \dots, x_{m-1}) = -\left(P_{m,0}(x)\right)_{m-1}(x_0, x_1, \dots, x_{m-1})$ in the columns $k-1, \dots, m-2-k$.

Thus row k is $O_{k-2}P_{m,1}X_1$ is the sub row x_{m-1} of length 1. Repeating this process produces a sequence of bottom rows of the form $O_{k-i-1}P_{m,i}X_i$ where X_i is some sub row of length i. At the end we have a bottom row $P_{m,k-1}X_{k-1}$ establishing our claim.

The final step is a sequence of elementary row operations to clear out the sub row x_{k-1}. Each such operation is realizable by pre multiplication by a matrix homotopic to the identity. This does not affect the rest of the last row.

This completes the proof of the homotopy commutativity of the diagram in theorem 2.

For theorem 1 we note that it is an immediate consequence of theorem 2 and proposition (2.3, ii).

4. Applications

1. The Whitney invariant of $\widetilde{\varnothing}_{\mathbf{m}}$. The adjoint of a nonsingular bilinear map $F_{m,m}$: $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{2m-1}$ defines a map $F'_{m,m} : S^{m-1} \to V_{2m-1,m}$ whose homotopy class is the Smale invariant of the symmetric immersion $\widetilde{F}_m : S^{m-1} \to \mathbb{R}^{2m-1}$ induced by $F_{m,m}$ on the diagonal sphere. For $F_{m,m} = \varnothing_{m,m}$ Lam's result on the homotopy class $[f_{m,m}]$ and the equivalence of Whitney and Smale invariants [3, §3] provide on some level an explanation of Brown's theorem for $\widetilde{\varnothing}_m$. Our claim was to provide an explicit regular homotopy from $\widetilde{\varnothing}_m$ to an immersion $\widetilde{\psi}_m$ in general position whose double point set would provide a visible proof of Brown's theorem for $\widetilde{\psi}_m$. As a start, there exist, by virtue of the cyclicity of $\pi_{m-1}(V_{2m-1,m})$ a homotopy $\widetilde{F}_m \sim i \circ H_{m,m}$ for some map $H_{m,m} : \ S^{m-1} \to \mathbb{R}^m \setminus \{0\}$ whose degree satisfies $\left[F'_{m,m}\right] = (deg\ F_{m,m})\, \mathrm{e}_{m-1}$ and via adjointness a regular homotopy $\widetilde{F}_m \sim \widetilde{H}_m$ where $\widetilde{\psi}_m(x) = (\ \mathrm{x}_{m-1}H_{m,m}(x),\ \mathrm{x}')$. In fact theorem 2 provides an explicit homotopy $f_{m,m} \sim i \circ P_{m,m-1}$ and hence a regular homotopy from $\widetilde{\varnothing}_m$ to the $\widetilde{\psi}_m$ defined by $H_{m,m}= P_{m,m-1}$. Unfortunately determining the double point set of $\widetilde{\psi}_m$ and deciding if it is in general position are not easily resolved questions ($m = 2, 3$ are exceptional cases), suggesting a further search for a more suitable representative in the regular homotopy class of $\widetilde{\varnothing}_m$.

Here is a family of immersions whose double point set is computable. Start with the n^{th} power map $f_n(z) = z^n$ on S^1 for $n \neq 0$. In real coordinates (a, b) this defines two polynomials $(Re(z^n),\ Im(z^n) = (\ p_n(a, b),\ \mathrm{q}_n(a, b))$ satisfying $p_n(a, b) = p_n(a, -b)$ and $\mathrm{q}_n(a, b) = -q_n(a, -b)$

Secondly define $G_{m,n} : \ S^{m-1} \to \mathbb{R}^m \setminus \{0\}$ by

$$G_{m,n}(x) = (x'',\ p_n\left(x_{m-2.}, x_{m-1}\right),\ p_n\left(x_{m-2}, x_{m-1}\right),\ q_n\left(x_{m-2}, x_{m-1}\right))$$

where x'' is obtained from x by omitting the last two components. Since $G_{m,n}$ is essentially the $(m-2)^{\text{nd}}$ suspension of $G_{2,n}$, $\deg G_{m,n} = n$. Observe that $G_{m,n}\left(x'', x_{m-2}, x_{m-1}\right) = -G_{m,n}\left(x'', x_{m-2}, -x_{m-1}\right)$ if and only if $x'' = 0$ and $p_n\left(x_{m-2.}, x_{m-1}\right) = 0$.

Finally define $\widetilde{G}_{m,n} : \ S^{m-1} \to \mathbb{R}^{m-2}$ to be the map

$$\widetilde{G}_{m,n}(x) = \left(x_{m-1}\ G_{m,n}(x),\ x'\right) \in \mathbb{R}^{2m-1} = \mathbb{R}^m \times \mathbb{R}^{m-1}$$

We omit the verification that $\widetilde{G}_{m,n}$ is an immersion, i.e. that rank $\widetilde{G}_{m,n} = m-1$ at all $x \in S^{m-1}$. From the above discussion the Whitney invariant of $\widetilde{G}_{m,n}$ equals the degree of $G_{m,n}$ *mod* 2 i.e. n, *mod* 2. From (4.1) and (4.2) we have that $\widetilde{G}_{m,n}(x)$ is a double point if and only if $x'' = 0$ and $G_{m,n}\left(x',\ x_{m-1}\right) = G_{m,n}\left(x',\ -x_{m-1}\right)$. The later condition holds exactly when $p_n\left(x_{m-2},\ x_{m-1}\right) = 0$. In terms of the rotation angle θ, this means $\cos(n\theta) = 0$ and hence $(x_{m-2},\ x_{m-1})$ is an n^{th} root of the imaginary number i or $-i$. These $2n$ points provide n pairs of double point preimages of $\widetilde{G}_{m,n}$, each pair consisting of two points with the same "real" parts. One can also check that $\widetilde{G}_{m,n}$ is in general position.

Thus $\widetilde{G}_{m,n}$ is regularly homotopic to $\widetilde{\varnothing}_m$ if and only if n is even and m is not of the form 2^p, or n is odd and $m = 2^p$ for some $p \geq 1$. Call an immersion of S^n in $\mathbb{R}^{2n}$ symmetric if there is a representative of its regular homotopy class which factor through $\mathbb{R}P^n$. We can restate this discussion as follows.

Proposition (4.1). $\widetilde{G}_{m,n}$ is a symmetric immersion if and only if n is even when m is not of the form 2^p or n is odd when $m = 2^p$ for some $p \geq 1$.

Remarks:

1) In a sense our theorem 2 replaces Lam's essentially homological determination of $[f_{m,m}]$ thereby supplying a polynomial representative $H_{m,m} = P_{m,m-1}$ and an explicit homotopy from $f_{m,m}$ to $P_{m,m-1}$.
2) The Whitney and Smale invariants of $\widetilde{\varnothing}_m$ are not all obvious from the definitions, but both are for $\widetilde{G}_{m,n}$.
3) Our computation of $\deg P_{m,m-1}$ is independent of SW class considerations which seem closely tied to previous accounts of Brown's theorem.

2. Brown's Theorem [1], [2], [5]. The following is a modification of the Smith's proof [5]

Lemma (4.2). If n is odd, the normal bundles of all symmetric immersions are isomorphic. If n is even, the Euler numbers of all symmetric immersions are equal *mod* 4.

Proof. For any symmetric immersions $f_1,\ f_2 : S^n \to \mathbb{R}^n$ we can find classifying maps for their normal bundles of the form

$$S^n \quad \to \quad \mathbb{R}P^n \quad \overset{v_i}{\to} \quad \mathrm{BO}(n) \ \ i = 1,\ 2$$

which also satisfy

1) v_1 is constant in a neighborhood of a closed disk D^n; and
2) v_1 agrees with v_2 in a neighborhood of $\mathbb{R}P^n/D^n$. Then v_2 is homotopic to the composite

$$\mathbb{R}P^n \quad \overset{\mathrm{p}}{\to} \quad \mathbb{R}P^n \bigvee S^n \quad \overset{v_1 \bigvee h}{\to} \quad \mathrm{BO}(n)$$

where p is the map that pinches the boundary of D^n to a point and h is the classifying map of some stably trivial bundle over S^n. But then this produces a classifying map for the normal bundle of f_2 of the form

$$S^n \quad \overset{\mathrm{p}}{\to} \quad S^n \bigvee S^n \bigvee S^n \quad \overset{v_{1\pi} \bigvee h \bigvee h}{\to} \quad \mathrm{BO}(n)$$

For n even, then the Euler numbers satisfy the equation $\mathcal{E}\,(f_2) = \ \mathcal{E}\,(f_1) + 2\mathcal{E}(h)$. The assertion follows since $\mathcal{E}(h)$ I even. For n odd the map $h \bigvee h :\ S^n \bigvee S^n \longrightarrow\ BO(n)$ is nullhomotopic since $[h] \in ker\{\pi_n(\mathrm{BO}(n)) \longrightarrow \pi_n(\mathrm{BO}(n+1))\} \cong 0$ *or* $\mathbb{Z}/2\mathbb{Z}$ according as $n \in \{1,\ 3,\ 7\}$ or not. This completes the proof.

Corollary (4.3). Let $n \neq 1,\ 3,\ 7$. Then all symmetric immersions of S^n in $\mathbb{R}^{2n}$ have the same Whitney invariant *mod* 2.

Proof. This follows from the fact that the Whitney invariant is characterized by the normal bundle [2], [10].

Theorem (4.4) [2]. The Whitney invariant of a symmetric immersion of S^n in $\mathbb{R}^{2n}$ is 1 *mod* 2 *if* $n = 2^p - 1$ *and* 0 *mod* 2 *otherwise*.

Proof. If $n \neq 2^p - 1$ (resp. $n = 2^p - 1$) then by theorem 1 the Whitney invariant of the symmetric immersion $\varnothing_{n+1}$ is 0 (*resp.* 1) *mod* 2, so the corollary above gives the result for all n, except $n = 1,\ 3,\ 7$. For these cases we refer the reader to the previous proofs.

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