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Mathematics

A note on certain multiplicative arithmetic functions

Una nota sobre ciertas funciones aritméticas multiplicativas

R. Sivaraman [Φ] **and J. López-Bonilla**[Ψ1]
[Φ] **Department of Mathematics, Dwaraka Dossw Goverdhan Doss Vaishnav College, Chennai 600 106, Tamil Nadu, India**
[Ψ] **ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México**

Abstract. We use the Apostol-Robbins theorem to determine $\sum_{d|n} (-1)^d f(\frac{n}{d})$ for several arithmetic functions of interest in number theory

Keywords. Liouville function; Euler's totient function; Sum of divisors function; Möbius function.

Resumen. Empleamos el teorema de Apostol-Robbins para determinar $\sum_{d|n} (-1)^d f(\frac{n}{d})$ para diversas funciones aritméticas de interés en teoría de números.

Palabras Claves. Funciones de Liouville; Totient de Euler; Función Suma de divisores; Función de Möbius.

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1. Introduction

We have the following result for any multiplicative arithmetic function [1–3]:

$$\sum_{d|n} (-1)^d f\left(\frac{n}{d}\right) = \left[\sum_{j=0}^{k-1} f\left(2^j\right) - f\left(2^k\right)\right] \sum_{d|m} f(d), \qquad n = 2^k\, m, \quad k \geq 0, \quad m \ \ is\ odd, \ (1)$$

which we shall apply to several functions of importance in number theory; (1) is compatible with the value $f(1) = 1$. In fact, the present article is an extension of the calculations made in [3].

2. Applications of (1)

(a) $f(n) = I(n) = n$ [4], then (1) implies the property [2, 5]:

[1] Corresponding author: jlopezb@ipn.mx

$$\sum_{d|n} (-1)^{d-1} \frac{n}{d} = \sum_{d|m} d = \sigma_O(n) = \sigma(m), \tag{2}$$

where $\sigma_O(n)$ is the sum of all odd divisors of n, that is, the sum of all divisors of m [6]

(b) $f(n) = \varphi(n)$ = Euler's totient function (n) [1, 4, 7] therefore (1) gives the following Liouville's result [2, 4, 8]:

$$\sum_{d|n} (-1)^d \varphi\left(\frac{n}{d}\right) = \begin{cases} -m, & k = 0, \\ 0, & k \geq 1, \end{cases} \tag{3}$$

hence:

$$\sum_{even\ \ d|n} \varphi(\frac{n}{d}) = \sum_{odd\ \ d|n} \varphi(\frac{n}{d}), \qquad k \geq 1. \tag{4}$$

(c) For the sum of divisors function and the divisor function [4], we obtain the expressions:

$$\sum_{d|n} (-1)^d \sigma\left(\frac{n}{d}\right) = -(k+1) \sum_{d|m} \sigma(d)\ , \sum_{d|n} (-1)^d d(\frac{n}{d}) = \frac{1}{2}\ (k+1)(k-2) \sum_{d|m} d(d)\ . \tag{5}$$

(d) $f(n) = \lambda(n)$ = Liouville's function (n) [4, 9], then from (1):

$$\sum_{d|n} (-1)^d \lambda\left(\frac{n}{d}\right) = \begin{cases} \frac{1}{2}\left(1 - 3(-1)^k\right), & if\ \ m\ \ is\ a\ square, \\ 0, & otherwise, \end{cases} \tag{6}$$

thus:

$$\sum_{even\ \ d|n} \lambda(\frac{n}{d}) = \sum_{odd\ \ d|n} \lambda(\frac{n}{d}), \qquad m\ \neq square. \tag{7}$$

(e) $f(n) = d^*(n)$ = Number of unitary divisors of n [4, 8], hence (1) gives the relation:

$$\sum_{d|n} (-1)^d d^* \left(\frac{n}{d}\right) = \begin{cases} -\ d\left(m^2\right), & k = 0, \\ (2k-3)\ d\left(m^2\right), & k \geq 1. \end{cases} \tag{8}$$

(f) $f(n) = \mu(n)$ = Möbius function (n) [1, 4, 10–13], then (1) implies the property:

$$\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right) = \begin{cases} -\ e_0(m), & k = 0, \\ 2\ e_0(k)\ e_0(m), & k \geq 1. \end{cases} \tag{9}$$

(g) $f(n) = \mu^*(n)$ = Unitary analogue of the Möbius function [4, 8], thus from (1):

$$\sum_{d|n} (-1)^d \mu^* \left(\frac{n}{d}\right) = \begin{cases} -\ \sum_{d|m}\ \mu^*(d), & k = 0, \\ \\ (3-k)\ \sum_{d|m}\ \mu^*(d), & k \geq 1. \end{cases} \tag{10}$$

(h) $f(n) = \mu(n)\ d(n)$:

$$\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right) d\left(\frac{n}{d}\right) = \begin{cases} 3\ (-1)^{\omega(m)}\ , & k = 1, \\ -(-1)^{\omega(m)}\ , & k \neq 1, \end{cases} \tag{11}$$

where $\omega(m)$ is the number of distinct prime factors of m.

(i) $f(n) = \mu(n)\ \sigma(n)$:

$$\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right) \sigma\left(\frac{n}{d}\right) = A(k)\ (-1)^{\omega(m)}\ \gamma(m), \qquad A(k) = \begin{cases} -1, & k=0, \\ 4, & k=1, \\ -2, & k \geq 2, \end{cases}, \tag{12}$$

where $\gamma(m)$ denotes the product of all prime factors of m.

(j) $f(n) = \lambda^{-1}(n) = \mu(n)\ \lambda(n)$:

$$\sum_{d|n} (-1)^d \mu\left(\frac{n}{d}\right) \lambda\left(\frac{n}{d}\right) = \begin{cases} -\ 2^{\omega(m)}, & k=0, \\ 0, & k=1, \\ 2^{\omega(n)}, & k \geq 2. \end{cases} \tag{13}$$

(k) $f(n) = (\mu(n))^2$:

$$\sum_{d|n} (-1)^d \left(\mu\left(\frac{n}{d}\right)\right)^2 = B(k)\ 2^{\omega(m)}, \qquad B(k) = \begin{cases} -1, & k=0, \\ 0, & k=1, \\ 2, & k \geq 2. \end{cases} \tag{14}$$

(l) $f(n) = \Lambda(n)$: von Mangoldt function [1, 4, 10, 14]:

$$\sum_{d|n} (-1)^d \Lambda\left(\frac{n}{d}\right) = C(k) \ln m\ , \qquad C(k) = \begin{cases} -1, & k=0, \\ 1 + (k-2)\ln 2\ , & k \geq 1\ . \end{cases} \tag{15}$$

(m) $f(n) = e(n) = 1$ [4] :

$$\sum_{d|n} (-1)^d = (k-1)\ d(m), \qquad k \geq 0. \tag{16}$$

(n) $f(n) = \frac{(\mu(n))^2}{\varphi(n)}$ [15] :

$$\sum_{d|n} (-1)^d \frac{\left(\mu\left(\frac{n}{d}\right)\right)^2}{\varphi\left(\frac{n}{d}\right)} = B(k)\ \frac{m}{\varphi(m)}\ . \tag{17}$$

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