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Mathematics

Recurrence relations and Jacobi theta functions

Relaciones de recurrencia y funciones theta de Jacobi

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Abstract. We consider two recurrence relations for the number of representations of an integer as sums of squares, and we show that the corresponding coefficients into such recurrences can be written in terms of theta functions.

Keywords. Sum of divisors function; Sequences of integers; Sums of squares; Theta functions; Recurrence relations.

Resumen. Consideramos dos relaciones de recurrencia para el número de representaciones de un entero como sumas de cuadrados, y mostramos que los correspondientes coeficientes en tales recurrencias pueden escribirse en términos de funciones theta.

Palabras Claves. Función suma de divisores; Secuencias de enteros; Suma de cuadrados; Funciones theta; Relaciones de recurrencia.

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1. Introduction

In [1] is proved the following recurrence relation for $r_k(n)$, the number of representations of n as a sum of k squares [2–4]:

$$n\ r_k(n) = 2k \sum_{j=1}^{n} A(j) r_k(n-j), \tag{1}$$

such that:

$$A(j) = (-1)^{j-1}\ j \sum_{odd\quad d|j} \frac{1}{d}\ , \tag{2}$$

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then is easy to obtain the values

$$A(1)=1,\quad A(2)=-2,\quad A(3)=4,\quad A(4)=-4,\quad etc., \tag{3}$$

thus it appears the sequence:

$$\begin{gathered}1,-2,4,-4,6,-8,8,-8,13,-12,12,-16,14,-16,\\ 24,-16,18,-26,20,-24,32,-24,24,-32,31,-28,\ldots\end{gathered} \tag{4}$$

In Sec. 2 we study the sequence (4) and its connection with the Jacobi theta functions [5–7].

2. Sequence (3) in terms of theta functions

We visited the On-line Encyclopedia of Integer Sequences [8] and for our surprise we find that (4) is the sequence A186690, therefore:

$$-\frac{1}{8}\,\frac{\vartheta_3''(0,\ q)}{\vartheta_3(0,\ q)}=\sum_{n=1}^{\infty}A(n)q^n=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\ \ j\ q^j}{1\ -\ q^{2j}}\ , \tag{5}$$

that is, the coefficients into (1) are connected to the Jacobi theta functions via (5).

On the other hand, Prof. Michael Somos [9] indicates the following relation between sequences:

$$A186690\ (n)=(-1)^{n-1}\ A002131\ (n), \tag{6}$$

hence:

$$A(n)=\begin{cases}-\left(\sigma(n)-\sigma\left(\frac{n}{2}\right)\right), & n\ is\ even,\\ \sigma(n), & n\ is\ odd,\end{cases} \tag{7}$$

involving the sum of divisors function $\sigma(n)$ [10–12], which implies the result:

$$\frac{n}{2}\sum_{k=1}^{n}\frac{(-1)^{n-k}}{k}\binom{n}{k}r_k(n)=\begin{cases}\sigma(n)-\sigma\left(\frac{n}{2}\right), & n\ is\ even,\\ \sigma(n), & n\ is\ odd,\end{cases} \tag{8}$$

that is, a connection between the sum of divisors function and the number of representations of an integer as a sum of squares.

The following recurrence relation was showed in [13]:

$$\begin{gathered}r_k(n)=2\sum_{j=0}^{n}(-1)^jB(j)\ r_{k+1}(n-j),\ B(n)=-\textstyle\sum_{j=1}^{n}\ (-1)^jb(j)\ B(n-j),\\ n\geq 1,\quad B(0)=\frac{1}{2}\ ,\end{gathered} \tag{9}$$

where:

$$b(j)=\begin{cases}2\ , & n=m^2\ ,\quad m\ \geq 1,\\ 1, & n=0\ ,\\ 0, & otherwise,\end{cases} \tag{10}$$

thus for $n=1,\ 2,\ 3,\ 4,\ 5,\ 6,\ldots.$:

$$B(n)=1,\ 2,\ 4,\ 7,\ 12,\ 20,\ 32,\ 50,\ 77,\ 116,\ 172,\ 252,\ 364,\ldots\ , \tag{11}$$

then from [8] we find that (11) is the sequence A014968, therefore:

$$\frac{1}{2}\left(\frac{1}{\vartheta_4(q)}-1\right)=\sum_{n=1}^{\infty}B(n)q^n, \tag{12}$$

in terms of a Jacobi theta function.

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