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Euler-Lagrange algebraic equations for polynomials representing a trajectory

Ecuaciones algebraicas de Euler-Lagrange para polinomios representando una trayectoria

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Abstract. In this paper, considering a 1-dimensional physical system and Weierstrass's polynomial approximation theorem, a polynomial version of Hamilton's principle is constructed, in which, instead of considering numerous continuous curves that connect two fixed points, polynomials are considered representative of these curves. As a result of the mathematical development, a set of algebraic equations (Euler-Lagrange) are discovered, whose solutions do not correspond directly to the trajectory of the considered particle, but to the independent coefficients of a polynomial that represents this trajectory.

Keywords. Hamilton's Principle; Weierstrass Theorem; Euler-Lagrange

Resumen. A partir de un sistema físico unidimensional y se utiliza el teorema de aproximación polinomial de Weierstrass para construir una versión polinomial del Principio de Hamilton, en la cual en lugar de analizar las numerosas curvas continuas conectando dos puntos fijos, se emplean polinomios representando a dichas curvas. Como resultado de este desarrollo matemático, se obtiene un conjunto de ecuaciones algebraicas (Euler-Lagrange) cuyas soluciones corresponden directamente a los coeficientes independientes de un polinomio que representa a la trayectoria de la partícula bajo estudio.

Palabras Claves. Principio de Hamilton; Teorema de Weierstrass; Euler-Lagrange

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1. Introduction

A simplified version of the central aspect in Hamilton's formulation of Classical Mechanics [1–4] could be obtained by substituting mathematical concepts and tools such as: functional, extremal of a functional, differential equation, etc. by the most basic concepts and tools of: real functions, polynomials, algebraic equation etc., while preserving the objective of this formulation. This simplification is achieved by implementing Weierstrass's polynomial approximation theorem[5]; in other words, the explicit incorporation of polynomials as representatives of the curves considered in the context of Hamilton's Principle makes it possible to "replace" the variational treatment by a treatment (less effective but simplified) in terms of real functions and vectors, as we will see in following sections.

On the other hand, an approach based on Weierstrass's theorem was used earlier in the construction of some quantum propagators [7]. As part of the solution to the problem considered here, we obtain a set of algebraic equations whose solutions do not correspond directly to the trajectory traversed by the considered particle (solution of the Euler-Lagrange equation), but to the values of the independent coefficients of a representative polynomial of this trajectory.

2. The polynomials

Let's restrict ourselves to the simplest case with only one spatial coordinate, z. The considered polynomials have the following form,

$$Z(t) = \sum_{k=0}^{n} \gamma_k F(t)^k, \tag{1}$$

where $\{\gamma_k\}$ represents the set with $n+1$ numerical coefficients initially free; $F(t)$ corresponds to the value of a function F and t symbolizes the time variable.

[5] The possibility of finding a polynomial approximation of a continuous function, defined in a finite interval, with any prescribed degree of accuracy was first demonstrated by Weierstrass [5, 6].

Consistent with the approach considered, of matching polynomials with curves, all passing through the same extreme points P_0 and P, being,

$$P_0 \ = \ (z_0, T_0), \qquad P \ = \ (z, T), \tag{2}$$

fixed in 2-dimensional spacetime, the polynomials in (1) should be required to check,

$$Z(T_0) = z_0, \qquad Z(T) = z. \tag{3}$$

From the independent requirements in (3) it follows that two coefficients γ_k, in expression (1), cannot be kept free; we choose γ_0 and γ_1 as these coefficients.

From (1) and (3) we can write the following expressions,

$$Z(T_0) = \gamma_0 + \sum_{k=1}^{n} \gamma_k F(T_0)^k = z_0, \tag{4}$$

$$Z(T) = \gamma_0 + \gamma_1 F(T) + \sum_{k=2}^{n} \gamma_k F(T)^k = z. \tag{5}$$

To determine γ_0 and γ_1 from (4) and (5) there is no need to define an $F(t)$ specific; however, for simplicity, we will fix their values at the instants T_0 and T, in particular, we will assume that,

$$F(T_0) = 0, \qquad F(T) = 1. \tag{6}$$

So, from (6), (5) and (4) we have,

$$\gamma_0 = z_0, \qquad \gamma_1 = z - z_0 - \sum_{k=2}^{n} \gamma_k, \tag{7}$$

where the independent coefficients in (1) correspond to the values $\{2, 3, ..., n\}$ of the index k. For convenience, we will identify the set of free coefficients (or independent) by a vector $\vec{\gamma} = (\gamma_2, ..., \gamma_n)$, with $n-1$ real components; thus, we write (1) highlighting this identification,

$$Z(\vec{\gamma}, t) \ = \ z_0 \ + \ (z - z_0)F(t) \ + \ \sum_{k=2}^{n} \gamma_k \Big(F(t)^k - F(t) \Big). \tag{8}$$

3. Polynomial version of Hamilton's principle

The version we started to develop corresponds to a polynomial approach for Hamilton's principle based on the (polynomial approximation) theorem of Weierstrass. We deduce a set of algebraic equations "equivalent" to the Euler-Lagrange equation in the sense that the solution of this set determines all the independent numerical coefficients of a polynomial that represents the path physically traversed by the considered particle (for a given order n of these polynomials). To derive these equations, we followed the usual procedure [2, 3], but making the necessary adaptations.

In the $z-t$ plane, with z being the third spatial coordinate and t the time variable, we represent with the letter q a reference curve (which we will assume to be the extremal of the action) that connects two fixed points,

$$P_0 \;=\; \Big(T_0, q(T_0)\Big), \qquad P \;=\; \Big(T, q(T)\Big). \tag{9}$$

Under the context under consideration, the curve q corresponds to several polynomials[6] with the form (1) and with coefficients that assume specific values. Let us consider one of these polynomials, whose independent coefficients are gathered in a vector $\vec{\gamma}$; that is, we have the correspondence,

$$\vec{\gamma} \quad \longleftrightarrow \quad q. \tag{10}$$

In a similar way, we consider an arbitrary curve η, "neighbor" to q, which connects the same extreme points P_0 and P, and which has an associated polynomial with independent coefficients gathered in a vector $\vec{\xi}$; that is, we have the correspondence,

$$\vec{\xi} \quad \longleftrightarrow \quad \eta. \tag{11}$$

To mathematically characterize the neighborhood between the curves η and q, and to represent the fact that η is arbitrary, the following identification must be correct,

$$\vec{\xi} \;=\; \vec{\gamma} \;+\; h\vec{\sigma}, \tag{12}$$

where h is a parameter that can take on a small value and $\vec{\sigma}$ is an arbitrary vector.

[6] Many polynomials can represent (approximately) the same curve.

The value of the function[7] (action) S for the polynomial corresponding to the curve q would write

$$S(\vec{\gamma}) = \int_{T_0}^{T} d\tau \mathcal{L}(\vec{\gamma}, \tau). \tag{13}$$

The directional derivative of the function (action) S along the direction of the arbitrary vector $\vec{\sigma}$ is defined here by $D_{\vec{\sigma}} S(\vec{\gamma})$, by the expression,

$$D_{\vec{\sigma}} S(\vec{\gamma}) = \lim_{h \to 0} \left(\frac{S(\vec{\gamma} + h\vec{\sigma}) - S(\vec{\gamma})}{h} \right). \tag{14}$$

From (13) we can write,

$$S(\vec{\gamma} + h\vec{\sigma}) - S(\vec{\gamma}) = \int_{T_0}^{T} d\tau \Big(\mathcal{L}(\vec{\gamma} + h\vec{\sigma}, \tau) - \mathcal{L}(\vec{\gamma}, \tau) \Big). \tag{15}$$

Then we develop the function $\mathcal{L}(\vec{\gamma} + h\vec{\sigma}, \tau)$, in the integrand of (15), in a Taylor expansion around $\vec{\gamma}$ until the first order term in h, that is,

$$\mathcal{L}(\vec{\gamma} + h\vec{\sigma}, \tau) - \mathcal{L}(\vec{\gamma}, \tau) = h\vec{\sigma} . \frac{\partial \mathcal{L}}{\partial \vec{\gamma}} + O(h^2), \tag{16}$$

where $\partial \mathcal{L} / \partial \vec{\gamma}$ is a gradient with respect to the independent variables γ_k. Note that in (16) we do not have a term of the type,

$$h\, \dot{\vec{\sigma}} . \left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{\gamma}}} \right),$$

because the vectors $\vec{\gamma}$ do not depend on time, which is a characteristic that we will use later. Then, we can write,

$$\frac{S(\vec{\gamma} + h\vec{\sigma}) - S(\vec{\gamma})}{h} = \int_{T_0}^{T} d\tau \left(\vec{\sigma} . \frac{\partial \mathcal{L}}{\partial \vec{\gamma}} + O(h) \right). \tag{17}$$

Thus, from (14) and (17), we find the following expression,

$$D_{\vec{\sigma}} S(\vec{\gamma}) = \int_{T_0}^{T} d\tau \; \vec{\sigma} . \frac{\partial \mathcal{L}}{\partial \vec{\gamma}}. \tag{18}$$

[7] Representing the curves by polynomials, that is, by the vectors defined from the coefficients, the "action" functional becomes a dependent function of "$n-1$" independent variables.

Then, according to the Extremal Principle, the derivative $D_{\vec{\sigma}}S(\vec{\gamma})$ must be zero for every vector $\vec{\sigma}$, which leads to writing,

$$\frac{\partial \mathcal{L}}{\partial \vec{\gamma}} = \vec{0} \quad \Longrightarrow \quad \frac{\partial \mathcal{L}}{\partial \gamma_k} = 0 \ , \quad k \in \{2, ..., n\}. \tag{19}$$

Expression (19) represents a set of $n - 1$ algebraic equations, with n arbitrary, but fixed, which we will call "Euler-Lagrange algebraic equations". By solving these equations, the corresponding independent coefficients can be found to a polynomial (among many), formally given in (8), because so far $F(t)$ is not defined, which represents the trajectory covered by the considered physical system.

4. Lagrangian polynomial for a free particle

Let us consider the specific case of a particle of mass m in a 1-dimensional free motion along a spatial direction identified here with the z coordinate direction of a given reference frame. The trajectory of this particle is represented, consistent with Weierstrass' theorem, by a time-dependent polynomial, $Z(t)$, with the form (8), and the corresponding Lagrangian function is defined by the expression,

$$\mathcal{L}(\dot{Z}) \ = \ \frac{1}{2} m \dot{Z}^2, \tag{20}$$

where $\dot{Z}$ represents the time derivative of Z. Using (8), we rewrite (20) as follows,

$$\begin{aligned} \mathcal{L}(\vec{\gamma}, t) \ &= \ \frac{m}{2}\left((z - z_0) \ + \ \sum_{k=2}^{n} \gamma_k \Big(kF(t)^{k-1} - 1 \Big) \right) \times \\ &\times \left((z - z_0) \ + \ \sum_{l=2}^{n} \gamma_l \Big(lF(t)^{l-1} - 1 \Big) \right) \dot{F}(t)^2. \end{aligned} \tag{21}$$

After having identified that in expression (21), when developed explicitly, there are two simple sums with the same value, we arrive at the expression,

$$\begin{aligned} \mathcal{L}(\vec{\gamma}, t) \ &= \ \frac{m}{2} \sum_{k=2}^{n} \sum_{l=2}^{n} \gamma_k \gamma_l \Big(klF(t)^{k+l-2} \ - \ 2kF(t)^{k-1} + 1 \Big) \dot{F}(t)^2 \ + \\ &+ m(z - z_0) \sum_{k=2}^{n} \gamma_k \Big(kF(t)^{k-1} - 1 \Big) \dot{F}(t)^2 \ + \ \frac{m}{2}(z - z_0)^2 \dot{F}(t)^2. \end{aligned} \tag{22}$$

Observing (22) it can be observed that if the function F, which until now is free, is assigned values defined by the expression,

$$F(t) \ = \ (t-T_0)/(T-T_0), \tag{23}$$

which checks conditions (6), then the last term in (22) corresponds directly to the kinetic energy of the free particle, that is, to the standard Lagrangian of this particle. Note that expression (22) itself corresponds to a polynomial version of a non-standard Lagrangian for the free particle.

Above an interesting aspect of this polynomial approximation and the definition of F is revealed. It can be observed that the coefficients for which the first two terms in (22) cancel out reduce this Lagrangian to the standard; therefore, these coefficients directly provide information on a polynomial that can be associated with the particle's trajectory. So, to determine the values of these coefficients we do,

$$\sum_{k=2}^{n}\sum_{l=2}^{n}\gamma_k\gamma_l\Big(klF(t)^{k+l-2}-2kF(t)^{k-1}+1\Big)+2(z-z_0)\sum_{k=2}^{n}\gamma_k\Big(kF(t)^{k-1}-1\Big) \ = \ 0, \tag{24}$$

Making $\gamma_k=0$, for $2\leq k\leq n$, with n arbitrary, we have that each one of the terms in (24) cancels, separately, so that (24) is identically verified. For these values we have, from (7), that,

$$\gamma_0=z_0, \qquad \gamma_1=z-z_0, \tag{25}$$

Thus, expression (8), considering (23), is written as,

$$Z(t)=\gamma_0+\gamma_1\left(\frac{t-T_0}{T-T_0}\right)=z_0+(z-z_0)\left(\frac{t-T_0}{T-T_0}\right) \tag{26}$$

from where we have,

$$Z(t)-z_0=\left(\frac{z-z_0}{T-T_0}\right)(t-T_0). \tag{27}$$

which clearly represents a straight line, in the $z-t$ plane, passing through the points (z_0,T_0) and (z,T). Therefore, the set of coefficients $\gamma_0=z_0$, $\gamma_1=z-z_0$ and $\gamma_k=0$, with $2\leq k\leq n$, define a polynomial representative of the rectilinear trajectory of the 1-dimensional free particle.

Next, we paid attention to the following mathematical facts: (i) According to the Hamilton's principle, we must consider a multitude of curves (here polynomials) continuously differentiable that join two fixed space-time points; (ii) the last term in (22) is independent of polynomials. Based on these comments, we will take provisionally as a "polynomial Lagrangian" for the free particle 1-dimensional no longer (22), but the one that results from excluding its last term; or be,

$$\mathcal{L}(\vec{\gamma},t) = \frac{m}{2(T-T_0)^2}\sum_{k=2}^{n}\sum_{l=2}^{n}\gamma_k\gamma_l\left(kl\left(\frac{t-T_0}{T-T_0}\right)^{k+l-2} - 2k\left(\frac{t-T_0}{T-T_0}\right)^{k-1} + 1\right) + + \frac{m(z-z_0)}{(T-T_0)^2}\sum_{k=2}^{n}\gamma_k\left(k\left(\frac{t-T_0}{T-T_0}\right)^{k-1} - 1\right). \tag{28}$$

where, to generate the polynomials corresponding to the different curves (including the classical trajectory) considered in Hamilton's principle, we must take the coefficients γ_k as real variables that can assume arbitrary values.

5. Euler-Lagrange algebraic equations for the 1-dimensional free particle

So far, the following characteristics can be noted in the previous results:
(I) Equations (19) are homogeneous.
(II) The coefficients γ_k are independent of time, and,
(III) The polynomials considered, with the structure (8), depend on time through the choice made in (23), that is, $F(t) = (t-T_0)/(T-T_0)$.

From the characteristics indicated above, we can conclude that the Lagrangians in the "Euler-Lagrange algebraic equations" should not contain linear terms in the coefficients, as these would generate, from (28), non-homogeneous equations; which, in turn, would imply that the coefficients that solve these equations would necessarily be time-dependent, contradicting one of the characteristics inherent to the approximation considered here. Therefore, we must reduce, once again, the Lagrangian in (28) to the following "Effective Lagrangian",

$$\mathcal{L}(\vec{\gamma},t) = \frac{m}{2(T-T_0)^2}\sum_{k=2}^{n}\sum_{l=2}^{n}\gamma_k\gamma_l\left(kl\left(\frac{t-T_0}{T-T_0}\right)^{k+l-2} - 2k\left(\frac{t-T_0}{T-T_0}\right)^{k-1} + 1\right) \tag{29}$$

which is adequate to the general context of the considered polynomial approximation and to the algebraic equations found.

Next, just for simplicity, let's fix the degree of the polynomials taking $n = 3$. In the general case, with arbitrary n, we would have to explicitly write $(n-1)^2$ terms when developing the double sums in (29). In the case considered, the Lagrangian (29) is written explicitly as follows,

$$\mathcal{L}(\vec{\gamma}, t) = \frac{m}{2}\dot{F}^2\left[\left(4F^2 - 4F + 1\right)\gamma_2^2 + \left(9F^4 - 6F^2 + 1\right)\gamma_3^2\right] + \\ + \frac{m}{2}\dot{F}^2\left[2\left(6F^3 - 3F^2 - 2F + 1\right)\gamma_2\gamma_3\right]. \quad (30)$$

where the symbol F is given by expression (23). Next, we must calculate the derivatives. We have,

$$\frac{\partial \mathcal{L}}{\partial \gamma_2} = m\dot{F}^2\left(4F^2 - 4F + 1\right)\gamma_2 + m\dot{F}^2\left(6F^3 - 3F^2 - 2F + 1\right)\gamma_3,$$

and considering (19) we have,

$$\Longrightarrow \quad \left(4F^2 - 4F + 1\right)\gamma_2 + \left(6F^3 - 3F^2 - 2F + 1\right)\gamma_3 = 0, \quad (31)$$

And,

$$\frac{\partial \mathcal{L}}{\partial \gamma_3} = m\dot{F}^2\left(9F^4 - 6F^2 + 1\right)\gamma_3 + m\dot{F}^2\left(6F^3 - 3F^2 - 2F + 1\right)\gamma_2,$$

and considering (19) we write,

$$\Longrightarrow \quad \left(6F^3 - 3F^2 - 2F + 1\right)\gamma_2 + \left(9F^4 - 6F^2 + 1\right)\gamma_3 = 0, \quad (32)$$

Equations (31) and (32) can be rewritten as follows,

$$(2F-1)^2\gamma_2 + (2F-1)(3F^2-1)\gamma_3 = 0, \quad (33)$$

$$(2F-1)(3F^2-1)\gamma_2 + (3F^2-1)^2\gamma_3 = 0. \quad (34)$$

whence it is clear that (34) can be obtained from (33) by multiplying it by $(3F^2 - 1)/(2F - 1)$; therefore, we only have one independent equation for two unknowns, so there are infinitely many solutions, which is generally expected due to the mathematical fact that we can assign many polynomials to a specific curve between two given points. We must remember the complementary requirement here given in item II above. This is fundamental, because if we ignore it, we could have coefficients in equations (33) and (34) that would depend on time through F; for example,

$$\gamma_2 = -\frac{(3F^2 - 1)}{(2F - 1)}\,\gamma_3,$$

Thus, the only admissible solution for (33) and (34) is to take: $\gamma_2 = \gamma_3 = 0$. But, we have already seen, in section 4, that these values correspond, precisely, to a polynomial that represents a rectilinear trajectory between the points P and P_0, as seen in (27).

6. Conclusion

(I) The possibility of using polynomials, as an approximation for curves, was identified in the context of Hamilton's variational principle, appealing to the Weierstrass approximation theorem. (II) The incorporation of this polynomial approximation in the context of Hamilton's principle made it possible to find a set of algebraic equations whose solution directly provides the independent coefficients of a polynomial that represents the corresponding physical trajectory. (III) The polynomial approximation that we have presented carries a significant limitation: it is not possible to represent all the continuously differentiable curves that join the given extreme points, as required by Hamilton's Principle, through polynomials with the form (8), with fixed degree. (IV) For the sake of internal mathematical consistency, the requirement arose that the Lagrangians to be considered in the algebraic equations must not include first-order terms in the coefficients. (V) We consider the case of Euler-Lagrange algebraic equations for a 1-dimensional free particle and verify that their solutions correspond to a correct polynomial representation for its trajectory. (VI) Our approach is structurally different from other polynomial approximations that can be found in the literature [8, 9].

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