

Partial Proof of a Conjecture with Implications for Spectral Majorization

Demostración parcial de una conjetura con implicaciones para la mayorización espectral

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Abstract. In this paper we report on new results relating to a conjecture regarding properties of $n \times n$, $n \leq 6$, positive definite matrices. The conjecture has been proven for $n \leq 4$ using computer-assisted sum of squares (SoS) methods for proving polynomial nonnegativity. Based on these proven cases, we report on the recent identification of a new family of matrices with the property that their diagonals majorize their spectrum. We then present new results showing that this family can be extended via Kronecker composition to $n > 6$ while retaining the special majorization property. We conclude with general considerations on the future of computer-assisted and AI-based proofs.

Keywords. Computer-assisted proof methods, AI, diagonalizable positive definite, diagonal majorization, IRGA, majorization, polynomial nonnegativity, positive-definite similarity, PD-diagonalizable, RGA, special PD-diagonalizable, spectrum majorization, special matrices, sum of squares (SoS).

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1. Introduction

In this paper we report new developments based on the proven cases of a surprising conjecture relating to special properties of $n \times n$ positive definite (PD) matrices for $n \leq 6$. It is argued in [8] that traditional mathematics has focused primarily on results that hold generally for all n , whereas most theoretical physics models, and most applied mathematics and engineering problems, are intrinsically defined in a fixed (and small) number of dimensions, e.g., time and the three spatial dimensions of ordinary experience. What is not commonly recognized is that as soon the dimensionality of a problem becomes fixed, e.g., to 3 dimensions, opportunities exist to establish potentially useful properties of the system of interest that do not hold generally in higher dimensions. Unfortunately, proving such properties typically requires computer-assisted methods that are not familiar to most scientists, mathematicians, and engineers.

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In Section 2, we illustrate these statements by showing how the IRGA conjecture for $n \leq 3$ is relatively straightforward to establish by hand, while proof of the $n = 4$ case was accomplished using powerful computer-assisted proof methods. We discuss why such methods are likely required, and why proofs for the remaining $n = 5$ and $n = 6$ cases are likely beyond the capabilities of current state-of-the-art methods on even the most powerful supercomputers.

In Section 3, we describe how proven cases of the IRGA conjecture define a fixed-dimensional family of matrices for which the diagonal majorizes the spectrum [9]. In Section 4, we present new results showing that Kronecker products of these matrices retain this unique majorization property. Then, in Section 5, we conclude with considerations on the imminent arrival of AI-based theorem provers, which can be viewed as proof oracles rather than computer-assisted tools.

2. IRGA Conjecture

Given a real symmetric $n \times n$ positive-definite (PD) matrix P , the author conjectured the following:

IRGA Conjecture: Given any real $n \times n$ symmetric positive-definite (PD) matrix P , then for $n \leq 6$:

$$\bar{S} = (P \circ P^{-1})^{-1} \in \text{nonsingular nonnegative PD doubly-stochastic} \quad (1)$$

where “ \circ ” is the Schur-Hadamard elementwise matrix product.

As an example, given that the following PD matrix P

$$P = \begin{bmatrix} 1.93758 & 1.04850 & -2.92314 & 0.83685 \\ 1.04850 & 1.29729 & -1.49559 & 0.56616 \\ -2.92314 & -1.49559 & 8.97170 & -0.55271 \\ 0.83685 & 0.56616 & -0.55271 & 0.61107 \end{bmatrix} \quad (2)$$

the conjecture implies that the following must be (and is) doubly-stochastic:

$$(P \circ P^{-1})^{-1} = \begin{bmatrix} 0.34750 & 0.13498 & 0.25568 & 0.26184 \\ 0.13498 & 0.58077 & 0.11800 & 0.16625 \\ 0.25568 & 0.11800 & 0.47034 & 0.15597 \\ 0.26184 & 0.16625 & 0.15597 & 0.41593 \end{bmatrix}. \quad (3)$$

The form $M \circ M$, for general nonsingular matrix M , is a familiar tool in the field of process control and is referred to as the Relative Gain Array (RGA) [1, 3, 7], and its inverse inspires the IRGA Conjecture appellation. Two key properties of the RGA are invariance with respect to arbitrary diagonal scaling, i.e., $\text{RGA}(M) = \text{RGA}(DME)$ for nonsingular diagonal matrices D and E , and a sum of 1 for entries in each row and column [1, 3, 5]. Both of these properties can also be verified to hold for $(P \circ P^{-1})^{-1}$; so proof of the conjecture can be reduced to proving that $(P \circ P^{-1})^{-1}$ is nonnegative, which implies that the result is doubly stochastic.

We note that a proof of the conjecture for $n = k$ immediately subsumes the $n < k$ cases. The $n = 1$ case is trivial, and the 2×2 case is also amenable to hand verification. Proof of the $n = 3$ case, however, requires a more structured approach involving a parameterization of PD matrices. This can be done in terms of the lower-triangular Cholesky square root, $\text{Chol}(P) \cdot \text{Chol}(P)^T = P$, which is guaranteed to exist for any PD matrix. In the case of P for $n = 3$, the Cholesky square root can be expressed as:

$$\text{Chol}(P) = \begin{bmatrix} d_1 & 0 & 0 \\ x & d_2 & 0 \\ y & z & d_3 \end{bmatrix} \quad (4)$$

Because the form of the conjecture is invariant with respect to left and right multiplication by a positive diagonal matrix D , by taking $D = \text{diag}(1/d_1, 1/d_2, 1/d_3)$, we can assume without loss of generality that the Cholesky square root of interest has the form

$$L_3 = \text{Chol}(DPD) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}, \quad (5)$$

which now has three fewer variables. Letting $R = L_3 L_3$, the conjecture implies $S = (R \circ R^{-1})^{-1}$ is doubly stochastic. Because a permutation similarity can move any off-diagonal entry of S to any other off-diagonal position, it is now sufficient by typicality to show that any off-diagonal entry of S is nonnegative. It can be verified that the polynomial for entry (2,3) of S is:

$$a^2b^2 + abc + c^2 + a^2c^2 + b^2c^2 - 2abc^3 + a^2c^4 \quad (6)$$

which, although somewhat arduous, can be transformed by hand into sum-of-squares (SoS) form:

$$(ac)^2 + \frac{1}{4} \left(ab\sqrt{2-\sqrt{3}} + c\sqrt{2+\sqrt{3}} \right)^2 + \quad (7)$$

$$\frac{1}{4} \left(ab\sqrt{2+\sqrt{3}} + c\sqrt{2-\sqrt{3}} \right)^2 + (bc - ac^2)^2, \quad (8)$$

which establishes nonnegativity of the chosen typical entry, hence also the nonnegativity and doubly stochasticity of S .

Unfortunately, the degree of the resulting polynomial for a typical entry of grows rapidly with n , and it can be verified that the corresponding entry from $S = (L_4 L_4^T)$ is:

$$\begin{aligned} & - (d^2 + e^2 + f^2 + 1) (a (f^2 + 1) (b^2 + c^2 + 1) (-ac^2 f^2 - ac^2 + 2acef - ae^2 - a + bcf^2 + bc \\ & - bef - cdf + de - b(ab+c) (-cf^2 - c + ef) (acf^2 + ac - aef - bf^2 - b + df) - d(-acf + ae + bf - d) \\ & ((f^2 + 1) (- (b^2 + c^2 + 1)) (ad + e)(cf - e) - f(ab + c) (-cf^2 - c + ef) (bd + ce + f)) \\ & - f(bd + ce + f) (-af(bd + ce + f) (-ac^2 f^2 - ac^2 + 2acef - ae^2 - a + bcf^2 + bc - bef - cdf + de) \\ & - b(ad + e)(cf - e) (acf^2 + ac - aef - bf^2 - b + df) \end{aligned} \quad (9)$$

for which the conversion to SoS form is not only well-beyond what can feasibly be done by hand, it is beyond the practical capabilities of most if not all general-purpose computer algebra systems (CAS), e.g., Mathematica. To give a feel for the rapid growth in complexity going from $n = 3$ to $n = 4$, the following is the Cholesky parameterization

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{bmatrix} \quad (10)$$

from which the polynomial for entry (1,2) of S_4 is:

$$\begin{aligned}
 S_4(1,2) = & \\
 & - (d^2 + e^2 + f^2 + 1)(a(f^2 + 1)(b^2 \\
 & + c^2 + 1)(-ac^2f^2 - ac^2 + 2acef \\
 & - ae^2 - a + bcf^2 + bc - bef - cdf + de) \\
 & - b(ab + c)(-cf^2 - c + ef) \\
 & (acf^2 + ac - aef - bf^2 - b + df)) \\
 & - (ad + e)(cf - e)(d(f^2 + 1)(-b^2 \\
 & + c^2 + 1))(-acf + ae + bf - d) \\
 & - bf(bd + ce + f)(acf^2 + ac - aef \\
 & - bf^2 - b + df)) - f(bd + ce + f)(d(-(ab \\
 & + c))(-cf^2 - c + ef)(-acf + ae + bf - d) \\
 & - af(bd + ce + f)(-ac^2f^2 - ac^2 + 2acef \\
 & - ae^2 - a + bcf^2 + bc - bef - cdf + de)).
 \end{aligned}$$

The SoS solution from [8], obtained using specialized state-of-the-art SoS methods [6, 10–12], consists of the following 25 squared terms:

$$\begin{aligned}
 p_1 &= 2(acdf - abef)^2, \\
 p_2 &= \left(\frac{1}{2}af^2c^2 + \frac{ac^2}{2} - \frac{1}{2}bf^2c + \frac{dfc}{2} \right. \\
 &\quad \left. - aefc - \frac{bc}{2} + \frac{ae^2}{2} + a + \frac{bef}{2} - \frac{de}{2} \right)^2, \\
 p_3 &= \frac{3}{4} \left(\frac{1}{3}af^2c^2 + \frac{ac^2}{3} - \frac{1}{3}bf^2c + \frac{2}{3}aefc \right. \\
 &\quad \left. - \frac{bc}{3} - \frac{dfc}{3} - ae^2 + de - \frac{bef}{3} \right)^2, \\
 p_4 &= \frac{2}{3} \left(-af^2c^2 - ac^2 + bf^2c + bc + aefc - \frac{dfc}{2} - \frac{bef}{2} \right)^2, \\
 p_5 &= \frac{1}{2} (ac^2f^2 - bcf^2 + cdf - acef)^2, \\
 p_6 &= \frac{1}{2} (ac^2f^2 - bcf^2 + bef - acef)^2, \\
 p_7 &= \left(-\frac{1}{2}aefc^2 + \frac{1}{2}ae^2c \right. \\
 &\quad \left. - \frac{1}{2}af^2c + ac + \frac{1}{2}befc - \frac{dec}{2} + \frac{aef}{2} \right)^2, \\
 p_8 &= \frac{3}{4} \left(aefc^2 - ae^2c - \frac{1}{3}af^2c + dec - befc + \frac{aef}{3} \right)^2, \\
 p_9 &= \frac{2}{3} (aef - acf^2)^2, \\
 p_{10} &= \left(-\frac{1}{2}adf^2c^2 + \frac{1}{2}bdfc + \frac{1}{2}abe^2 + ab - \frac{bde}{2} \right)^2, \\
 p_{11} &= \frac{3}{4} (adf^2c^2 - bdfc - abe^2 + bde)^2,
 \end{aligned}$$

$$\begin{aligned}
p_{12} &= (adf - abf^2)^2, \\
p_{13} &= \left(\frac{1}{2}aec^2 - \frac{bec}{2} - \frac{afc}{2} + ae \right)^2, \\
p_{14} &= \frac{3}{4}(-aec^2 + bec - afc)^2, \\
p_{15} &= 2a^2c^2f^2, \\
p_{16} &= \left(\frac{1}{2}adc^2 - \frac{bdc}{2} + ad \right)^2, \\
p_{17} &= \frac{3}{4} \left(-adc^2 + bdc - \frac{2abf}{3} \right)^2, \\
p_{18} &= \frac{5}{3}a^2b^2f^2, \\
p_{19} &= a^2c^2d^2, \\
p_{20} &= 2 \left(\frac{1}{2}ac^2f^3 - \frac{1}{2}bcf^3 + \frac{1}{2}cdf^2 + \frac{1}{2}bef^2 - acef^2 \right. \\
&\quad \left. + \frac{1}{2}ac^2f + \frac{1}{2}ae^2f + af - \frac{bcf}{2} - \frac{def}{2} \right)^2, \\
p_{21} &= 2 \left(-\frac{1}{2}ac^2f^3 + \frac{1}{2}bcf^3 - \frac{1}{2}cdf^2 - \frac{1}{2}bef^2 \right. \\
&\quad \left. + acef^2 - \frac{1}{2}ae^2f + \frac{def}{2} \right)^2, \\
p_{22} &= \frac{1}{2}(bcf - ac^2f)^2, \\
p_{23} &= a^2c^2e^2, \\
p_{24} &= a^2b^2d^2, \\
p_{25} &= a^2b^2e^2.
\end{aligned}$$

The restriction $n \leq 6$ of the conjecture can be established from counterexamples that are easily found for $n=7$ (an explicit counterexample is given in [8]). Therefore, it remains only to prove the $n=6$ case, which of course subsumes the $n = 5$ case. Unfortunately, the size of the polynomial for a typical off-diagonal entry of S_6 vastly exceeds the capability of any known SoS solution method. Purely as a dramatic illustration, the polynomial for entry (1,2) of S_6 is provided in the Appendix.

3. SPPD Matrices

In this section we consider one of the surprising implications of the IRGA conjecture and its proven cases. Letting k be the largest integer for which the conjecture holds, we define the following matrix form:

Definition (Special PD-Diagonalizable): *A given $n \times n$ matrix $M = PDP^{-1}$, $n \leq k$, is defined to be special PD-diagonalizable if matrix D is diagonal and positive-definite matrix P is real symmetric.*

The motivation for this definition is the following:

Majorization Theorem: *The spectrum of a special PD-diagonalizable matrix is majorized by its diagonal.*

Majorization is an important theoretical and practical property that is exploited in a variety of problem domains because it provides a preorder relational operator for comparing the relative

disorder (entropy) of the distribution of entries of two vectors. Specifically, given two vectors $x, y \in \mathbb{R}^n$ with equal entry sums, y is said to *majorize* x , notated as $x \prec y$, if and only if there exists a set of permutation matrices P_j and probabilities p_j such that

$$x = \sum_j p_j P_j y. \quad (11)$$

This expression of x as a probabilistic sum of permutations of y can be thought of as indicating that the entry distribution of x is more *disordered* in a particular sense than that of y . An alternative definition is that $x \prec y$ if and only if

$$x = Sy, \quad (12)$$

where S is a doubly-stochastic matrix, i.e., nonnegative with every row and column sum equal to unity. This definition makes the probabilistic interpretation even more explicit in the form $Sy \rightarrow x$, where x can be interpreted a conservative stochastic evolutionary state of y . This interpretation can be made rigorous by the following known result:

$$x \prec y \implies H(x) \geq H(y) \quad (13)$$

where $H(\cdot)$ is the Shannon entropy, $H(z) \doteq -\sum_{i=1}^n z_i \log z_i$, where $z_i \log z_i$ is taken as 0 for $z_i=0$.

The definition in terms of a doubly-stochastic relationship also permits majorization to be generalized from real to complex (or hypercomplex) vectors [2].

We begin our proof of the majorization theorem by noting a known result that relates the spectrum vector of a diagonalizable matrix $M = AEA^{-1}$, where $\lambda(A) = \text{diag}(E)$ for diagonal eigenvalue matrix E , to its diagonal vector via RGA(A) as:

$$(A \circ A) \cdot \text{diag}(M) = \lambda(M) \quad (14)$$

In the case of nonsingular RGA(A), this relationship can be expressed as:

$$\text{diag}(M) = (A \circ A)^{-1} \cdot \lambda(M). \quad (15)$$

In the case of special PD-diagonalizable M diagonalized by a real positive definite P , this gives

$$\text{diag}(M) = (P \circ P)^{-1} \cdot \lambda(M) \quad (16)$$

$$= S \cdot \lambda(M) \quad (17)$$

where S is doubly stochastic. From Eq.(12) we can infer that the diagonal of a special PD-diagonalizable matrix majorizes its spectrum, thus proving the majorization theorem.

The significance of this result is that it establishes special PD-diagonalizable matrices as only the second known matrix class for which a majorization relationship exists between the spectrum and the diagonal. The only previously known class is the set of unitarily diagonalizable matrices $M = UEU^*$. The majorization property of this class can be directly verified by applying Eq. (14)

$$(U \circ U) \cdot \lambda(M) = \text{diag}(M). \quad (18)$$

and observing that the Hadamard product of a unitary matrix and its transposed inverse necessarily gives a doubly stochastic result. This shows that the spectrum of any unitarily diagonalizable matrix majorizes its diagonal. Thus, the set of special PD-diagonalizable matrices becomes the only known matrix class with the reverse property of the diagonal majorizing the spectrum.

4. Generalized PD-Diagonalizable (GPDD) Matrices

In [9], an application of special PD-diagonalizable (SPDD) matrices involving manifold search. The problem assumes a k -dimensional manifold of diagonal matrices over a family of objects for which generalized majorization defines a meaningful partial order, where the set of objects/points has been subjected to a real continuously deformed positive definite gauge-like similarity transformation. In other words, each point of the manifold (or lattice) is an SPDD matrix. The goal is to proceed from a point on the manifold and proceed to a local optimum of high or low spectral entropy. The following is a high-level description of the basic algorithm:

- (i) Initialize the search at a random point on the SPDD manifold, represented by the matrix A_0 with diagonal entries $\mathbf{d}_0 = [d_{01}, d_{02}, \dots, d_{0N}]^\top$.
- (ii) For the current point on the manifold, compute its neighbors on the manifold. The neighbors can be defined according to a specific neighborhood relation, e.g., within a certain distance along the manifold or connected through a given transformation.
- (iii) Among the neighbors, identify the neighboring matrix A_k with diagonal entries $\mathbf{d}_k = [d_{k1}, d_{k2}, \dots, d_{kN}]^\top$ such that \mathbf{d}_0 majorizes \mathbf{d}_k , i.e., determine whether there exists a doubly-stochastic transformation relating \mathbf{d}_0 and \mathbf{d}_k that satisfies the desired direction of majorization.
- (iv) If such a neighboring matrix A_k is found, update the current point on the manifold to the new point corresponding to A_k and its diagonal entries \mathbf{d}_k . Return to step 2 and continue the search.
- (v) If no neighboring matrix is found that satisfies the majorization condition, terminate the search. The current point on the manifold corresponds to a local minimum with respect to the majorization property.

This search algorithm is designed to exploit the majorization property of SPDD matrices to traverse the transformed manifold in a systematic manner to a point corresponding to high or low spectral entropy using only diagonal information, i.e., without need for a spectral decomposition at every point. However, the practical applicability of the approach is severely constrained by the upper bound $k \leq n$ size limitation of SPDD matrices.

We note that the proven SPDD majorization relationship exploits an explicit construction of a doubly-stochastic relationship between the diagonal and spectrum based on the IRGA conjecture. However, the fact that such a construction does not generally exist for $n > 6$ does not necessarily imply that the diagonal-spectrum majorization relationship does not hold in general. In other words, the constructive proof for all $n \leq 4$ could be interpreted as possible evidence that the result may in fact hold in general, i.e., while $(P \circ P^{-1})^{-1}$ may not always be doubly stochastic for $n > 6$ – *though it often is* – there could exist other structural properties of a general PD-diagonalizable matrix that ensures the existence of a doubly-stochastic relationship between its diagonal and spectrum. Such structure may even hold for diagonalizability by arbitrary complex-valued PD matrices. Empirical investigation may be able to provide sufficient evidence to justify a conjecture or, alternatively, empirical and/or analytical efforts may identify counterexamples that constrain such generalizations

One viable generalization involves expanding the set of special PD-diagonalizable (SPPD) matrices to include PD matrices P obtained as a Kronecker product, $P = P_1 \otimes P_2$, of SPPD matrices P_1 and P_2 such that the majorization property is preserved. For example, if P_1 is 4×4 , and P_2 is 3×3 , then P would have size 12×12 . We now show that this generalization does in fact extend the class of matrices for which the diagonal majorizes the spectrum. We now provide a theorem formalizing such a generalization:

Generalized PDD Theorem: *Kronecker products of SPPD matrices retain the SPDD majorization property.*

We begin with a relatively general lemma:

Diagonal-Spectrum Mapping Lemma: Let M_1 and M_2 be $n \times n$ matrices with $\text{diag}(M_1) = S_1 \lambda(M_1)$ and $\text{diag}(M_2) = S_2 \lambda(M_2)$. The Kronecker product $S_1 \otimes S_2$ maps $\lambda(M_1 \otimes M_2)$ to the diagonal of $M_1 \otimes M_2$.

We note that the eigenvalues of the Kronecker product of two matrices are given by the Kronecker product of their eigenvalues:

$$\lambda(M_1 \otimes M_2) = \lambda(M_1) \otimes \lambda(M_2), \quad (19)$$

and the diagonal of $M_1 \otimes M_2$ is given by the Kronecker product of the diagonals of M_1 and M_2 :

$$\text{diag}(M_1 \otimes M_2) = \text{diag}(M_1) \otimes \text{diag}(M_2). \quad (20)$$

Now, consider the product of $S_1 \otimes S_2$ and $\lambda(M_1 \otimes M_2)$:

$$(S_1 \otimes S_2) \lambda(M_1 \otimes M_2) = (S_1 \otimes S_2)(\lambda(M_1) \otimes \lambda(M_2)). \quad (21)$$

Using the Kronecker mixed product property, $A \otimes B \cdot C \otimes D = AC \otimes BD$, we can rewrite the equation as:

$$(S_1 \otimes S_2) \lambda(M_1 \otimes M_2) = (S_1 \lambda(M_1)) \otimes (S_2 \lambda(M_2)). \quad (22)$$

Since $\text{diag}(M_1) = S_1 \lambda(M_1)$ and $\text{diag}(M_2) = S_2 \lambda(M_2)$, the equation becomes:

$$\text{diag}(M_1) \otimes \text{diag}(M_2) = (S_1 \otimes S_2) \lambda(M_1 \otimes M_2). \quad (23)$$

Thus, we have verified that the Kronecker product $S_1 \otimes S_2$ maps $\lambda(M_1 \otimes M_2)$ to the diagonal of $M_1 \otimes M_2$. This implies the following corollary:

Corollary: *Diagonal-Spectrum Majorization* - If M_1 and M_2 each possesses the property that its diagonal majorizes its spectrum, then $M_1 \otimes M_2$ also possesses this property.

The proof is straightforward: the majorization property of the corollary implies there exist doubly stochastic matrices $S_1 \lambda(M_1) = \text{diag}(M_1)$ and $S_2 \lambda(M_2) = \text{diag}(M_2)$. From the Diagonal-Spectrum Mapping Lemma, the Kronecker product $S_1 \otimes S_2$ maps $\lambda(M_1 \otimes M_2)$ to the diagonal of $M_1 \otimes M_2$. Because the Kronecker product of doubly stochastic matrices is doubly stochastic, we can conclude that the diagonal of $M_1 \otimes M_2$ majorizes its spectrum.

We have established the key property of the GPDD Theorem, but technically it remains to be shown that the Kronecker product of positive definite matrices is positive definite:

Kronecker PDD Lemma: Given PD-diagonalizable matrices $M_1 = P_1 E_1 P_1^{-1}$ and $M_2 = P_2 E_2 P_2^{-1}$, where P_1 and P_2 are positive definite and E_1 and E_2 are diagonal, the Kronecker product $M_3 = M_1 \otimes M_2$ has the form $M_3 = P_3 E_3 P_3^{-1}$ for some positive definite P_3 and diagonal E_3 .

To prove this, consider the Kronecker product $M_3 = M_1 \otimes M_2$:

$$M_3 = (P_1 E_1 P_1^{-1}) \otimes (P_2 E_2 P_2^{-1}).$$

Using the Kronecker mixed product property, we have:

$$M_3 = (P_1 \otimes P_2)(E_1 \otimes E_2)(P_1^{-1} \otimes P_2^{-1}).$$

Let $P_3 = P_1 \otimes P_2$ and $E_3 = E_1 \otimes E_2$. Since P_1 and P_2 are positive definite, their Kronecker product P_3 is also positive definite. Furthermore, since E_1 and E_2 are diagonal, their Kronecker product E_3 is also diagonal.

Finally, we show that $(P_1^{-1} \otimes P_2^{-1})$ is the inverse of P_3 . Using the Kronecker mixed product property, we have:

$$(P_1 \otimes P_2)(P_1^{-1} \otimes P_2^{-1}) = (P_1 P_1^{-1}) \otimes (P_2 P_2^{-1}) = I_n \otimes I_n = I_{n^2}.$$

This implies that $(P_1^{-1} \otimes P_2^{-1})$ is the inverse of P_3 , so M_3 has the sought form $M_3 = P_3 E_3 P_3^{-1}$, where P_3 is positive definite and E_3 is diagonal.

In summary, we have established that a PD-diagonalizable matrix of any size divisible by 2 or 3 (the only primes less than 4) can be constructed with the property that its diagonal majorizes its spectrum. However, we can generalize this to construct examples for all n using block-diagonal constructions:

GPDD Construction Lemma: Nontrivial GPDD matrices (i.e., containing no blocks that are strictly diagonal) can be constructed for any value of $n > 1$.

This can be proven inductively using the values $\{2, 3, 4\}$, since the IRGA conjecture has been proven for $n \leq 4$. We begin by extending our set of base cases to include 5 and 6 by noting that the case $n = 5$ can be constructed with nontrivial diagonal blocks of size 2 and 3, and the case $n = 6$ can be constructed with nontrivial blocks of size 2 and 4. Now we can assume the statement holds for all values of k such that $5 \leq k \leq n$. The inductive step for $n + 1$ can be established as follows:

- (i) If n is the sum of a sequence containing a 2, replace the 2 with a 4 to obtain $n + 2$. For $n + 1$, we can use the sequence for n and append a 2.
- (ii) If n is the sum of a sequence containing a 3, replace the 3 with a 4 to obtain $n + 1$. For $n + 2$, we can use the sequence for $n + 1$ and append a 2.
- (iii) If n is a sum of all 4s, then for $n + 1$ and $n + 2$, we can append a 2 or 3 respectively.

Thus, we can construct an instance of a block diagonal GPDD matrix, with every block of size greater than 1 (i.e., no trivial blocks), for any $n > 1$. Note that this does not imply that every $n \times n$ matrix $M = PDP^{-1}$ satisfies the Majorization Theorem, just that nontrivial instances which do satisfy it can be constructed for all $n > 1$.

5. Discussion

In this paper we have reported on the recently-proven $n = 4$ case of the IRGA conjecture, and how the conjecture has led to the discovery of the first matrix class for which the diagonal majorizes the spectrum. We then presented new results generalizing this class to include instances of any size n divisible by 2 or 3. More generally, this class includes *all* PD-diagonalizable matrices of size $n = 4$ and is conjectured (via the IRGA conjecture) to also include all PD-diagonalizable matrices of size 5 and 6. Some of these results are unlikely to ever have been recognized without computer assistance, and it is likely that the proofs for some could/can only be obtained by computer-assisted methods.

There seems little doubt that artificial intelligence (AI) agents will possess capabilities to solve mathematical problems vastly beyond what can be solved by any human. However, the results at the core of this paper motivate a conjecture that humans will be able to continue contributing to mathematical discovery. This idea is based on the fact that an arbitrarily long traversal of successive branches of derivation from an obscure result to some distant and seemingly unrelated result is easy, but identifying the path backward from that end-result alone can be vastly more difficult. With that, we conjecture the following:

Human vs. AI Conjecture: While an AI will be able mine a vastly larger expanse of conjectures than humans, and it will be able to prove vastly more of what it finds than humans, the infinitude of the expanse will leave overlooked seams from which new results can be extracted by humans that otherwise might never be found.

Because of its form, the conjecture can never be proven, but evidence for it can potentially be obtained from the following experiment:

Experiment: Give an AI (or a team of AIs) access to everything that humans know at the current time about mathematics, but with the exception of a carefully selected human-produced theorem, and its related results, and test whether the AI can prove the theorem.

The outcome of the experiment may seem obvious based on the anticipated power of future AIs, but consider the case in which the removed theorem is something like the following:

For any diagonalizable real matrix M , there exists a matrix X in the neighborhood of M with the property that $\text{diag}(X)$ majorizes its spectrum $\lambda(X)$ according to the generalized definition that vector u majorizes v if and only if there exists doubly-stochastic S such that $u = Sv$. (The definition of “neighborhood” is not given so as to avoid giving away too much information, but it is well-defined, nontrivial, and not inconsistent with uses of the term elsewhere in the literature.)

The actual truth of the statement is unimportant for present illustrative purposes, so we just assume a human, or a human using computer-assisted tools, has proven it based on observations like those discussed in this paper. Key question: *Is it possible that insufficient information is given in the statement for an AI to identify a starting point that can lead it to a result that satisfies the statement?* Probably not, but can the answer be foreseen with certainty? The results described in this paper derive from the IRGA conjecture, which has little more significance than the features of a grain of sand that distinguish it from the billions of surrounding grains on a beach. Anyone who chooses to pick up a grain at random will be able to record its features and, with high probability, be the only person (AI or human) who will ever observe those features.

A pessimist might argue that the Human/AI conjecture assumes a future in which humans are relegated to being sifters of minutiae, celebrating their trivial discoveries like a child showing a shiny stone to friends. An AI might agree but add that the pessimist’s description actually applies to all accomplishments in human history. However, if the AI were to fail even one instance of the proposed test, then it must recognize that one or more human discoveries² may someday prove essential for deriving results that are deemed important by both AIs and humans. In other words, ultimate assessments of importance must be left to the judgment of future historians – *whether they be AI or human.*

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² Note that the situation with human mathematicians is not at all analogous to AI versus human players of chess or other games. It is more analogous to crypto mining in which (it is assumed) the more mining that is done, the more coin that will be found. In this view, although humans can be thought of as low-power computers, they may still contribute in proportion to that net power.

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Appendix A. The $n = 6$ Polynomial

What follows is the $(1, 2)$ polynomial entry of S_6 . This is provided here to dramatically illustrate the increase in complexity over the $n=3$ and $n=4$ polynomials given in Eqs. (6) and (9), respectively. We note that this is the maximally simplified form achievable with Mathematica. (As per Mathematica convention, adjacency of variables implies multiplication.)

The exponential complexity of the problem can be understood intuitively by considering the number of degrees of freedom that must be examined to confirm or disconfirm whether it can be transformed to a sum of squares form.

$$S_6(1,2) =$$

$$\begin{aligned}
& -k(-k+a)m+b-n-a-c+n+d-p-a-e-p-b-f+p+a-c-f+p+g-q-a-h-q-b+i+q+a-c-i-q-d-j+q+a-e-j+q+b-f-j-q-a-c-f-j-q) \\
& (-b+g+c+h+i)(-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j)((d^2+e^2+f^2+1)(q^2+j^2+j^2-2-p-q+j+p^2+1)((a+g+h)(b+k+c+m+n)(-n+f+p+i-q-f-j-q) \\
& (-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j)(-b+g+c+h+i)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q) \\
& (-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j))-(d+g+e+h+f+i+j)(-j-q^2+p-q-j)((a+d+e)(b+k+c+m+n)(-n+f+p+i-q-f-j-q) \\
& (-e-q^2+j^2+c-f-q^2-j^2-e-j^2+c-f+j^2+h-q^2-j-c-i-q^2+j+h-j-c-i-j-m-q+j+c-n-q-j+2-e-p-q-j-2-c-f-p-q-j-e-p^2+c-f-p^2-e+c-f+m-p-c-n-p-h-p-q+c-i-p-q)(-b+d+c+e+f) \\
& (a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& ((a+g+h)(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j) \\
& (-b+g+c+h+i)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j))-(d+g+e+h+f+i+j) \\
& (-j-q^2+p-q-j)((a+b+c)(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f^2-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1))+(d+k+e+m+f+n+p)(j-q-p)((a+b+c)(b+g+c+h+i)(-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j) \\
& (-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+g+h)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1))-(g^2+h^2+i^2+j^2+1)(q^2+1)((b+d+c+e+f)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& ((a+d+e)(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-e-q^2+j^2+c-f-q^2-j^2-e-j^2+c-f+j^2+h-q^2-j-c-i-q^2+j+h-j-c-i-j-m-q+j+c-n-q-j+2-e-p-q-j-2-c-f-p-q-j-e-p^2+c-f-p^2-e+c-f+m-p-c-n-p-h-p-q+c-i-p-q)(-b+d+c+e+f) \\
& (a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& ((a+d+e)(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j) \\
& (-b+d+c+e+f)(a+g+h)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j) \\
& (-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j))-(d+g+e+h+f+i+j)(-j-q^2+p-q-j)((a+b+c)(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1))+(d+g+e+h+f+i+j)(-j-q^2+p-q-j)((a+b+c)(b+d+c+e+f)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& (-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+g+h)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1))-(g^2+h^2+i^2+j^2+1)(q^2+1)((b+d+c+e+f)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& (-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1))+(d+g+e+h+f+i+j)(-j-q^2+p-q-j)((a+b+c)(b+d+c+e+f)(-f-q^2-j^2-f^2+j^2+q^2+j+i-j-n-q-j+2-f-p-q-j-f-p^2-f+n-p-i-p-q) \\
& (-c-j^2-f^2-c-p^2-f^2-c-j^2-q^2-f^2-c-f^2+2-c-j-p-q-f^2+e-j^2-f+e-p^2-f+e-j^2-q^2-f-h-j-q^2-f+2-c-i-j-q^2-f+e-h-j-f+2-c-i-j-f-m-p-f+2-c-n-p-f+j-m-q-f-2-c-j-n-q-f+h-p-q-f-2-c-i-p-q-f-2-e-j-p-q-f-c-i^2-c-n^2-c-i^2-q^2+h-i-q^2-e-i-j-q^2-c+h-i-e-i-j+m-n-e-n-p-i-m-q-h-n-q+2-c-i-n-q+e-j-n-q+e-i-p-q) \\
& (-b^2+c^2+1)(a+g+h)(-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1)))+(b+k+c+m+n)(-n+f+p+i-q-f-j-q)(-g(-g-q^2+a+h-q^2+b+i-q^2-a-c-i-q^2+d+j-q^2-a-e-j-q^2-b-f-j-q^2+a-c-f-j-q^2+k-q-a-m-q-b-n-q+a-c-n-q-d-p+q+a-e-p-q+b-f-p-q-a-c-f-p-q-g+a+h+b-i-a-c-i+d+j-a-e-j-b-f+j+a-c-f-j)((d^2+e^2+f^2+1)(q^2+j^2+j^2-2-p-q+j+p^2+1)((a+g+h)(b+k+c+m+n)(-n+f+p+i-q-f-j-q) \\
& (-h-q^2+c+i-q^2+e-j-q^2-c-f-j-q^2+m-q-c-n-q-e-p-q+c-f-p-q-h+c+i-e-j-c-f-j)(-b+g+c+h+i)(a+k+m)(-m+c+n+e-p-c-f+p+h-q-c-i-q-e-j+q+c-f-j-q)(-i-q^2+f+j-q^2+n-q-f-p-q-i+f-j))-(d+g+e+h+f+i+j)(-j-q^2+p-q-j)((a+d+e)(b+k+c+m+n)(-n+f+p+i-q-f-j-q) \\
& (-e-q^2+j^2+c-f-q^2-j^2-e-j^2+c-f+j^2+h-q^2-j-c-i-q^2+j+h-j-c-i-j-m-q+j+c-n-q-j+2-e-p-q-j-2-c-f-p-q-j-e-p^2+c-f-p^2-e+c-f+m-p-c-n-p-h-p-q+c-i-p-q)(j^2-f^2+p^2-f^2+j^2-q^2-f^2-2-j-p-q-f^2+e-j^2-f^2-j^2-q^2-f-2-i-j-q^2-f-2-i-j-f-2-n-p-f+2-j-n-q-f+2-i-p-q-f+i^2+n^2+i^2-q^2-2-i-n-q+1)))+(d+g+e+h+f+i+j)(-j-q^2+p-q-j)(d(-d-q^2+j^2+a+e-q^2+j^2+b+f-q^2-j^2-a-c-f-q^2-j^2-d+j^2+a+e-j^2+b-f-j^2-a-c-f-j^2+g-q^2-j-a-h-q^2-j-b-i
\end{aligned}$$

$j^2 m e - a f m e + a i j m e - d j^2 n e + b f j^2 n e - a c f j^2 n e - d n e + b f n e - a c f n e + g j n e - 2 a h j n e - b i j n e + a c i j n e + b i^2 p e - a c j^2 p e + b p e - a c p e - g i p e + a h i p e + a f h j p e + d i j p e - b f i j p e + a c f i j p e + a f h q e + d i q e - b f i q e + a c f i q e - b j q e + a c j q e - b f^2 m + a c f^2 m - b i^2 m + a c i^2 m - b f^2 j^2 m + a c f^2 j^2 m + d f j^2 m - b m + a c m + d f m + g i m - a h i m - f g j m + a f h j m - d i j m + 2 b f i j m - 2 a c f i j m + a h^2 n + a n - g h n + b h i n - a c h i n + d h j n - b f h j n + a c f h j n - a f h^2 p - a f p + f g h p - b f h i p + a c f h i p + b f^2 h j p - a c f^2 h j p - d f h j p + b f^2 h q - a c f^2 h q + b h q - a c h q - d f h q - a i q + a f j q) + k (c f^2 m a^2 + c i^2 m a^2 + c f^2 j^2 m a^2 - e f j^2 m a^2 + c m a^2 - e f m a^2 - h i m a^2 + f h j m a^2 + e i j m a^2 - 2 c f i j m a^2 + e^2 n a^2 + h^2 n a^2 + e^2 j^2 n a^2 - c e f j^2 n a^2 - c e f n a^2 - c h i n a^2 - 2 e h j n a^2 + c f h j n a^2 + c e i j n a^2 + n a^2 - f h^2 p a^2 - c e i^2 p a^2 - c e p a^2 - f p a^2 + e h i p a^2 + c f h i p a^2 - c f^2 h j p a^2 + e f h j p a^2 - e^2 i j p a^2 + c e f i j p a^2 - c f^2 h q a^2 - c h q a^2 + e f h q a^2 - e^2 i q a^2 + c e f i q a^2 - i q a^2 + c e j q a^2 + f j q a^2 - c f^2 k a - c i^2 k a - c f^2 j^2 k a + e f j^2 k a - c k a + e f k a + h i k a - f h j k a - e i j k a + 2 c f i j k a - b f^2 m - a b i^2 m - a - b f^2 j^2 m + d f j^2 m - a - b m + a + d f m + a + g i m - a - f g j m - a - d i j m + a + 2 b f i j m - a - 2 d e j^2 n + a + c d f j^2 n + a + b e f j^2 n - a - 2 d e n + a + c d f n + a + b e f n - a - 2 g h n + a - c g i n + a + b h i n + a + 2 e g j n - a - c f g j n + a + 2 d h j n - a - b f h j n - a - c d i j n - a - b e i j n + a + c d i^2 p a + b e i^2 p a + c d p a + b e p a + 2 f g h p a - e g i p a - c f g i p a - d h i p a - b f h i p a - c f^2 g j p a - e f g j p a + b f^2 h j p a - d f h j p a + 2 d e i j p a - c d f i j p a - b e f i j p a + c f^2 g q a + c g q a - e f g q a + b f^2 h q a + b h q a - d f h q a + 2 d e i q a - c d f i q a - b e f i q a - c d j q a - b e j q a + b f^2 k + b i^2 k + b f^2 j^2 k - d f j^2 k + b k - d f k - g i k + f g j k + d i j k - 2 b f i j k + d^2 n + g^2 n + d^2 j^2 n - b d f j^2 n - b d f n - b g i n - 2 d g j n + b f g j n + b d i j n + n - f g^2 p - b d i^2 p - b d p - f p + d g i p + b f g i p - b f^2 g j p + d f g j p - d^2 i j p + b d f i j p - b f^2 g q - b g q + d f g q - d^2 i q + b d f i q - i q + b d j q + f j q) + (-b j^2 f^2 + a c j^2 f^2 - b f^2 + a c f^2 + d j^2 f - a e j^2 f + d f - a e - f g j f + a h j f + 2 b i j f - 2 a c i j f - b i^2 + a c i^2 - b + a c + g i - a h - i d i j + a e i j) (k^2 + m^2 + n^2 + p^2 + q^2 + 1))))))$

