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On the non-separability of the Lanczos-Dirac Delta and a function presenting a property of this Delta

Sobre la inseparabilidad de la Delta de Lanczos-Dirac y una función que presenta una propiedad de esta Delta

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Abstract. In this article, we show two independent proofs, which use of different properties of the Lanczos-Dirac Delta, of which, the Delta cannot be separable. Furthermore, the conditions that should be imposed on an ordinary function are identified so that it presents the translation property of the Lanczos-Dirac Delta.

Keywords. Lanczos-Dirac Delta; Non-separability.

Resumen. En este artículo mostramos dos pruebas independientes, que utilizan diferentes propiedades, de que la Delta de Lanczos-Dirac no puede separarse. Además, se identifican las condiciones que se deben imponer a una función ordinaria para que presente la propiedad de traslación de la Delta de Lanczos-Dirac.

Palabras Claves. Delta de Lanczos-Dirac; Inseparabilidad.

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1. Introduction

The theory of distributions, or generalized functions, developed by Prof. L. Schwartz, presents the Dirac Delta in its proper mathematical context [1]. Historically, it was Prof. C. Lanczos was the first to use this Delta in Quantum Mechanics [2], before Prof. P. A. M. Dirac introduced it [3].

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In non-relativistic Quantum Mechanics, particularly in its version in momentum space [4], the usefulness of the Lanczos-Dirac Delta is evident, as well as in Classical Electrodynamics [5]. Many properties of the Lanczos-Dirac Delta can be found in the literature [1], however, some characteristics of it may not have been directly highlighted in it because no specific problem was previously identified that could direct our attention to it.

The expression of the Lanczos-Dirac Delta suggests that it is not separable. Here we show, in two alternative ways, that the Lanczos-Dirac Delta, $\delta(x-y)$, is not separable. Furthermore, we identify the requirements that an ordinary function should satisfy to present the translation property of the Lanczos-Dirac Delta, which would be important, for example, when it is necessary to have an identification criterion that allows determining which wave functions (for example, of the electron in the Hydrogen atom) could be linearly superposed to generate a function that is approximately of the Lanczos-Dirac Delta type.

2. The Lanczos-Dirac Delta is not separable

In this section we present two independent proofs that the Lanczos-Dirac Delta is not separable. The first proof is based on the use of Delta properties that are expressed in terms of integrals and the second, the most direct, is based on the use of the Lanczos-Dirac Delta symmetry property. We warn that the first proof is informal, in the sense that Delta is treated as an ordinary function and that writing the integral in (1) corresponds, strictly speaking, to an arrangement that moves away from the formal expression [6], but here, however, it is convenient.

[I] Consider a real function f of a real variable and the translation property of the Lanczos-Dirac Delta:

$$f(x)=\int_{-\infty}^{\infty}\delta(y-x)f(y)dy, \tag{1}$$

Provisionally assuming that Delta is separable, we write,

$$\delta(y-x) \;=\; u(y)\,v(x). \tag{2}$$

Substituting (2.2) into (2.1), we have:

$$f(x) \;=\; v(x)\int_{-\infty}^{\infty}u(y)f(y)\,dy, \tag{3}$$

Multiplying expression (3) by the value of the function u evaluated at y', which could assume any value, including $y=x$, we have:

$$\begin{aligned} u(y')f(x) \;&=\; u(y')\,v(x)\int_{-\infty}^{\infty}u(y)f(y)\,dy,\\ u(y')f(x) \;&=\; \delta(y'-x)\int_{-\infty}^{\infty}u(y)f(y)\,dy, \end{aligned} \tag{4}$$

Integrating (4) with respect to the variable x over the entire real line, we have,

$$u(y')\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{\infty}\delta(y'-x)dx\;\int_{-\infty}^{\infty}u(y)f(y)\,dy=\int_{-\infty}^{\infty}u(y)f(y)\,dy, \tag{5}$$

For expression (2.5) to be consistent, it must be fulfilled that:

$$u(y')=\text{constant},\quad \forall y'\;. \tag{6}$$

On the other hand, considering values $x = x^\star \neq y'$ in expression (2.4), we have:

$$u(y')f(x^\star) \ = \ 0, \tag{7}$$

If the function f (which has no relation to the functions u and v) is non-zero, and can assume null values only for a finite set of points x_i, then it must be fulfilled that the value constant in (6) is null:

$$u(y') = 0 \ , \ \ \forall y' \ . \tag{8}$$

This implies, taking x and y as being independent, that:

$$v(x) \ u(y') \ = \ 0 \ , \ \ \forall y' \ , \forall x \ , \tag{9}$$

but the left side in (9) is the Lanczos-Dirac Delta; that is, we arrive at:

$$\delta(y' - x) = 0, \ \ \forall y' \ , \forall x \ , \tag{10}$$

which is inconsistent. Thus, the Lanczos-Dirac Delta is not separable, contrary to what is assumed.

$[II]$ Let us accept that the Lanczos-Dirac Delta is separable, so it should have the following structure:

$$\delta(x - y) = \omega(x) \ \omega(y), \tag{11}$$

because the left side in (2.11) must remain intact when exchanging x for y and vice versa, since Delta is even.
Now, we give y, a fixed value, say y_0, so by (2.11):

$$\delta(x - y) = \omega(x) \ \omega(y_0) = 0 \quad \text{for} \quad x \neq y_0, \tag{12}$$

which implies that $\omega(x) = 0$, if $x \neq y_0$, but as y_0 is arbitrary then it follows that:

$$\omega(x) = 0, \quad \text{for everything } x. \tag{13}$$

therefore, from (2.11) and (2.13), it follows that $\delta(x - y) = 0$, for arbitrary x and y. Thus, the Lanczos-Dirac is not separable.

3. Function presenting a Lanczos-Dirac Delta type property

Let Ψ be a real function of real variable y. Let's consider the expression that, without formal rigor [6], is written as:

$$\int_{-\infty}^{\infty} dy \ \delta(x - y) \ \Psi(y) = \Psi(x). \tag{14}$$

and consider in the integrand in (14) a real function F, with a real variable, instead of the Lanczos-Dirac Delta, so that, if possible, the following expression is fulfilled:

$$\int_{-\infty}^{\infty} dy \ F(x - y) \ \Psi(y) = \Psi(x), \tag{15}$$

that is, that presents Delta's translation property. It is clear that F should satisfy certain specific conditions, which we will identify below.

Consider the following integral,

$$M = \int_{-\infty}^{\infty} dy\ F(x-y)\Psi(y). \tag{16}$$

Making the change of variable: $x - y = -\ z$, we write:

$$M = \int_{-\infty}^{\infty} dz\ F(-z)\Psi(z+x). \tag{17}$$

Next, the function Ψ is expanded in a Taylor series around a point x,

$$\Psi(z+x) = \Psi(x) + z\Psi^{'}(x) + \frac{z^2}{2}\Psi^{''}(x) + ... \tag{18}$$

Thus, we get,

$$M = \Psi(x)\int_{-\infty}^{\infty} dzF(-z)\ +\ \Psi^{'}(x)\int_{-\infty}^{\infty} dzF(-z)z\ +$$
$$+\ \Psi^{''}(x)\int_{-\infty}^{\infty} dzF(-z)\frac{z^2}{2}\ +\ ... \tag{19}$$

so that, if F checks each of the following relationships:

$$\int_{-\infty}^{\infty} dzF(-z) \approx 1, \quad \int_{-\infty}^{\infty} dzF(-z)z \approx 0, \quad \int_{-\infty}^{\infty} dzF(-z)\frac{z^2}{2} \approx 0, \quad ... \tag{20}$$

we arrive at the expected result:

$$\int_{-\infty}^{\infty} dy\ F(x-y)\ \Psi(y) \approx \Psi(x), \tag{21}$$

which would show that F "behaves" like a Lanczos-Dirac Delta in the sense that it carries its translation property.

4. Conclusion

In section 2 of this article, two independent proofs were presented that the Lanczos-Dirac Delta cannot be separable. In section 3, we identify the requirements that should be imposed on an ordinary function so that it presents the translation property of the Lanczos-Dirac Delta.

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