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Statistics - Time Series

Hypothesis testing with explosive time series. An approach to the theory of the functional central limit theorem

Pruebas de hipótesis con series explosivas. Una aproximación a la teoría del teorema del límite central funcional

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Abstract. Dealing with uncertainty about the true data generating process requires a differentiated perspective of the distributions in hypothesis testing. In particular, the realizations, or the observed data, generated by interactions that are naturally ordered in time, posits a need for a differentiated analysis with respect to the standard statistics available for hypothesis testing. The Functional Central Limit Theorem provides a framework that enables the researcher to build a statistic that fits his data and hypothesis at hand. In this paper I show some of the necessary conditions under which the popular $t-statistic$ properly condenses the information of the underlying distribution as well as the additional tools available when then t distribution is not suitable for hypothesis testing.

Keywords. Limiting distributions; hypothesis testing; dynamic structures.

Resumen. Tratar con la incertidumbre sobre el verdadero proceso generador de datos requiere una perspectiva diferenciada con respecto a las distribuciones usadas en la pruebas de hipótesis. En particular, las realizaciones, o los datos observados, producto de las interacciones que están naturalmente ordenadas en el tiempo, plantean la necesidad de un análisis diferenciado con respecto a los estadísticos de prueba estándar disponibles para la prueba de hipótesis. El Teorema del Límite Central Funcional proporciona un marco que permite al investigador construir un estadístico que se ajuste a sus datos y a la hipótesis en cuestión. En este documento, muestro algunas de las condiciones necesarias bajo las cuales el popular estadístico$-t$ condensa adecuadamente la información de la distribución subyacente, así como las herramientas adicionales disponibles cuando la distribución t no es adecuada para la prueba de hipótesis.

Palabras Claves. Convergencia en distribución; pruebas de hipótesis; estructuras dinámicas.

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1. Introduction

The use of limiting distributions in order to reject hypotheses is the main instrument for empirical scientific research. In most of the empirical work, the researcher relies on the Law of Large Numbers and the Central Limit Theorem as these, jointly with the Neyman-Pearson principle, provide a clear decision rule in hypothesis testing.

The t statistic for linear models with additive and independent stochastic unobserved disturbances is among the most exploited. Whenever the true model is linear and the estimators converge in probability to the population parameters, the usual $t - statistic$ follows a t distribution and converges in distribution to the normal. In the literature of food science for instance, [1] describe the contributions of oil constituents to overall antioxidant capacity of walnut oil using linear models and [2] in their medical research, identify associations between an obesity life style risk index, built on 6 obesogenic lifestyle risk factors, and an obesogenic index (BMI) estimating linear models with data on 17,040 adults. In the world of the research in medicine, [3] compare the means on the types of primary tumors and colorectal liver metastases characteristics, between patients with a desmoplastic histological growth pattern (DHGP) and patients without a DHGP using the Student's $t - test$.

Before relying on the $t - statistic$ for hypothesis testing, [4] analyze the dynamic dependence structure of their data in order to draw conclusions on the relationship between water supply and bioenergy production. Certain data which is naturally ordered in time, such as the quarterly Gross Domestic Product (GDP), may exhibit patterns of dependence that deserve a different analysis than cross-sectional and spatial data. Variables widely used in social sciences such as GDP, stock prices, inflation, and exchange rates are typical examples of random variables that have a sequential interpretation. The underlying data generating process of the GDP is plausibly conditional on both the unobserved state of nature and the known sequential order of time. Thus, analyzing the trajectory of an stochastic process resembling the data at hand (the data on GDP in the research of [4]), provide additional knowledge for hypothesis testing. Given the advantages and practical usefulness of the $t - statistic$ and asymptotic theory, in this paper I describe the necessary conditions for proper hypothesis testing in the context of linear models as well as the asymptotic tools available for hypothesis testing that are specific to information with time-series properties. Thus, this paper may help the reader in the construction of a test in applied work when the usual normal or t distributions are not suitable as a reference for decision rules.

This document is divided into five sections including the introduction. In section two, I review the methodology along with the literature on test statistics based on asymptotic theory. In sections three and four, I show applications and a discussion. Finally, in section five, I conclude.

2. Methodology

Linear models are widely used in empirical research and the OLS estimator has been intensely exploited. Nobel Price Richard Thaler run a linear regression of current earnings changes over current forecast bias in earnings and find statistical evidence of optimism and overreaction in financial analysts' forecasts [5]. Following [6], [7] use a two-stage least squares (2SLS) regression and find statistical evidence of a relationship between education and formal employment.

[8] estimate a set of dynamic models in an attempt to shed light on the relationships between aggregate Consumers' Expenditure, Non-durables and Disposable Income. Ten years later, [9] showed that the usual t tests are misleading when the researcher is dealing with a data generating process whose first and second moments are not constant in time or whose covariance with respect to a fixed lagged is not constant. Nowadays, we use the proper asymptotic distributions and critical values for unit root and cointegration tests as developed in [10] and [11] and is common knowledge among those dealing with time-series that for two series $x_t \sim I(d_x)$ and $y_t \sim I(d_y)$,

the combination $z_t = bx_t + cy_t \sim I(\max(d_x, d_y))$ as noted by Granger in 1981 [12].

The first popular tests for unit roots were developed in [13] and [14]. Recent extensions in [15], [16] and [17] are suitable to test for explosivity.

2.1. Linear Models

Consider the following model:

$$y_i = \mu + \theta x_i + \epsilon_i \tag{1}$$

for $i = 1, 2, 3..., n$, $n \in \mathbb{N}$, and $\epsilon_i \sim n.i.i.d(0, \sigma^2)$. Also, y_i and x_i are observable variables with x_i exogenous, and (μ, θ) are population parameters. The log-likelihood function is

$$L_n = \sum_{i=1}^{n} logf(y_i, \boldsymbol{\theta}) = -\frac{n}{2}log2\pi - \frac{n}{2}log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \mu - \theta x_i)^2 \tag{2}$$

where $\boldsymbol{\theta}$ is the vector of parameters and $f(\cdot, \cdot)$ is the density of a normal distribution with mean $\mu + \theta x_i$ and variance σ^2. From the first order conditions the maximum likelihood (ML) estimator for θ is

$$\hat{\theta} = \frac{\hat{Cov}(x_i, y_i)}{\hat{\sigma}_x^2} \tag{3}$$

where $\hat{Cov}(\cdot, \cdot)$ is the sample covariance function and $\hat{\sigma_x}$ is the sample variance function of x_i. $\hat{\theta}$ is asymptotically normal for $x_i \epsilon_i$ independent. That is

$$\frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{\sigma^2 \mathbb{E}(x_i^2)}} \rightarrow_d N(0, 1) \tag{4}$$

The above follows mainly from the Lindeberg-Levy Central Limit Theorem (CLT) which is pivotal to hypothesis testing.

Theorem (Linderberg-Levy CLT):

Take a sequence of i.i.d. random scalars $w_1, w_2, ..., w_n$ with finite and constant expected values and variances $\mathbb{E}(w_i) = \mu_w, \mathbb{V}(w_i) = \sigma_w^2, \sigma_w^2 \neq 0$, and the statistic $\bar{w}_n = \frac{1}{n}\sum_{i=1}^{n} w_i$, then

$$\sqrt{n}(\bar{w}_n - \mu_w) \rightarrow_d N(0, \sigma_w^2)$$

Intuitively, as n becomes larger (the sample size increases), $\sqrt{n}(\bar{w}_n - \mu_w)$ approximates the normal, and this is true for any initial distribution of w_i. To grasp the idea of a limiting distribution, consider the sequence $Z_1, Z_2, ..., Z_n$ with distributions $F_1, F_2, ..., F_n$ as well as the variable Z with distribution F. A sequence $\{Z_i\}_{i=1}^{n}$ converges in distribution to Z if $\lim_{n\to\infty} |F_n(z) - F(z)| = 0$ for any z and we write $Z_n \rightarrow_d Z$. Take the variable Z with distribution $F(z) = 1 - e^{-z}$ and the sequence $Z_n = (0, n], n > 0$ with distributions $F_n(z) = 1 - \left[1 - \frac{z}{n}\right]^n$. Notice that

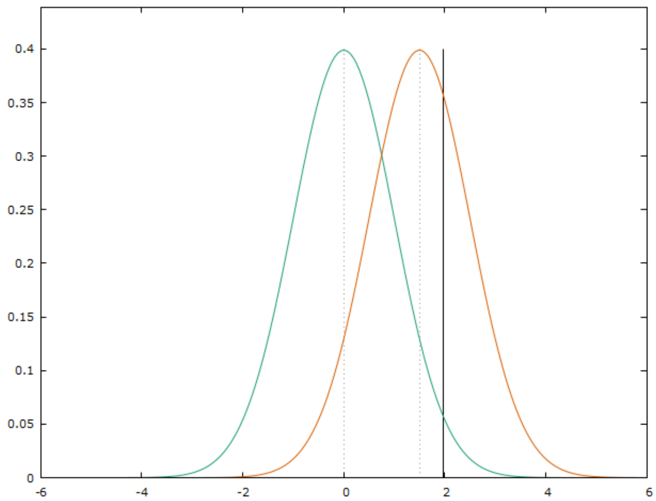
for $n = 1$: $Z_1 = (0, 1]$ and $F_1(z) = 1 - \left[1 - \frac{z}{1}\right]^1$
for $n = 2$: $Z_2 = (0, 2]$ and $F_2(z) = 1 - \left[1 - \frac{z}{2}\right]^2$
for $n = 3$: $Z_3 = (0, 3]$ and $F_3(z) = 1 - \left[1 - \frac{z}{3}\right]^3$
for $n \to \infty$: $\lim_{n\to\infty} z_n = (0, \infty]$ and $\lim_{n\to\infty} F_n(z) = 1 - e^{-z}$

that is, $Z_n \rightarrow_d Z$.

For equation 1 and the estimator expressed in 3, under the null hypothesis that $\theta = 0$, the $t - statistic$ is of the form:

$$t = \frac{\hat{\theta}}{\hat{\sigma_\theta}} \longrightarrow_d N(0,1) \tag{5}$$

For a well-specified linear model, the ML estimator is the same as the ordinary least squares estimator. Whenever the true distribution is of the exponential family, the ML estimator is consistent even when the the assumed distribution is not normal. The decision rule is then to reject the null hypothesis at the specified significance level, if the calculated $t - statistic$ is larger than the critical value. Notice that an estimator is a random variable itself. Hypothesis testing corresponds to taking a decision on whether the data at hand are drawings from a distribution with parameter θ_0 or drawings from a distribution with parameter θ_1. Thus, the researcher is always subject to errors: choosing θ_1 when θ_0 is true (type I error), choosing θ_0 when θ_1 is true (type II error). As the true parameters are not observable, one is always uncertain about which error is making. In this circumstances, the best we can do is to take decisions on the probabilities of choosing wrongly by setting a threshold on one of them. The researcher would prefer to have the largest power, or the highest probability of rejecting θ_0 when θ_0 is false, while he would like the probabilities of type I and type II errors to be at their lowest. Unfortunately, that type of optimization is not feasible so that usually we follow the Neyman - Pearson principle: we fix a value for one of the probabilities, usually the type I error probability or significance level, and then we minimize the other one.

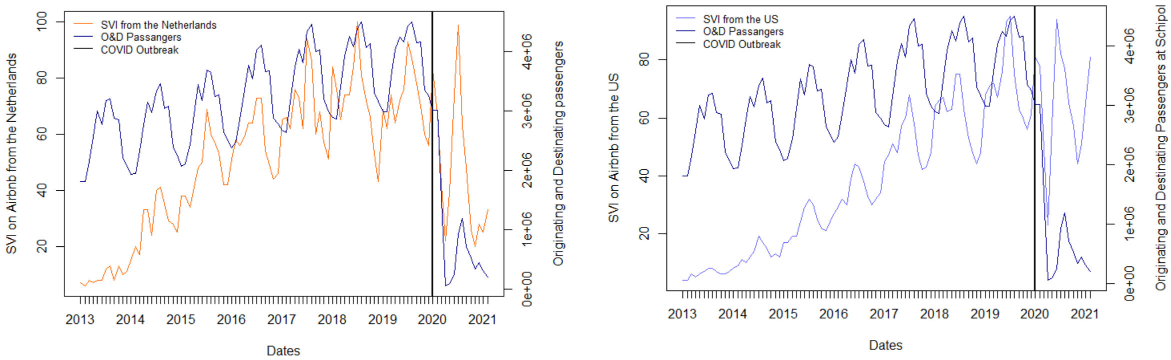


→ Null hypothesis → Alternative hypothesis → Critical Value --→ Expected Value

Figure 1: Alternative Decisions in Hypothesis Testing Under Normality

2.2. Dynamic Linear Models

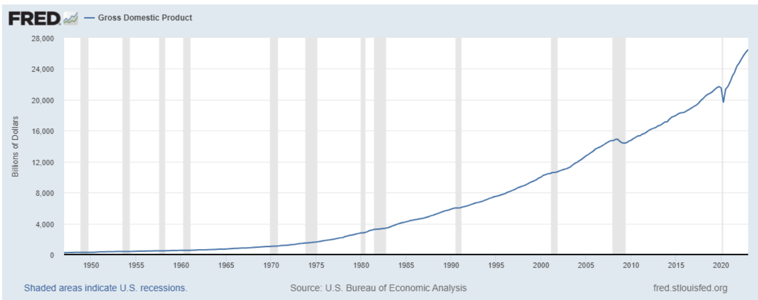
In many contexts, expressing the dynamics explicitly makes sense. Take, for example, the number of originating and destinating passangers at Schipol Airport. The time-series figure 2 suggests that the first moment of the series depends on the order of time. Panel a) shows the series of Google searches for the term "Schipol Airport" made from the Netherlands and panel b) shows the series for Google searches from the U.S. The dependence structure suggests that transit at the airport is related in time to Google searches and that it is worthwhile to statistically test this hypothesis.



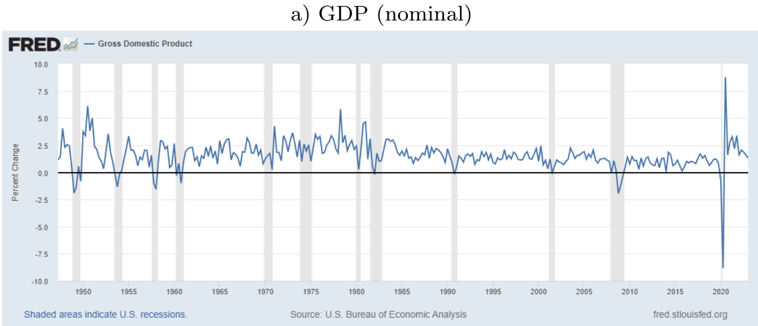
a) b)

Figure 2: Originating and Destinating Passengers at Schipol Airport

Figure 3 shows the quarterly series of the U.S. Gross Domestic Product (GDP): panel a) shows the series in levels and panel b) shows the first differences of the GDP scaled by the first lag of the GDP. While panel a) suggests that the first moment of the GDP is not constant, panel b) suggest that the first differences of the same information exhibit a constant expectation. In addition, both figures 2 and 3 suggest that the current values of the series are related to their past values. This suggestion is testable but there are not mechanics or physical descriptions of the social interactions that can bring a clear decision rule to accept or reject this hypothesis.



a) GDP (nominal)



b) GDP quarterly growth

Figure 3: U.S. Gross Domestic Product

Consider the data generating process $y_t = \rho y_{t-1} + u_t$, $\mathbb{E}(u_t) = 0$ and $\mathbb{V}(u_t) = \sigma^2$ for $t = 1, 2, 3..., T$. If the true parameter ρ equals one, that is $y_t = y_{t-1} + u_t$ and we use the estimator in expression 3 $\hat{\rho} = \dfrac{\sum_{t=1}^{T} y_{t-1}y_t}{\sum_{t=1}^{T} y_{t-1}^2}$ for hypothesis testing, the estimator $\hat{\rho}$ is not consistent for ρ and the corresponding $t-statistic$ $t = \dfrac{\hat{\rho}}{\hat{\sigma}_\rho}$ does not have a t distribution and its limiting distribution is not normal. Thus, the usual critical values do not convey relevant information for hypothesis testing. So far, we have two relevant unobserved worlds:

(i) $y_t = \phi + \rho y_{t-1} + \varepsilon_t$ *and* $\rho < 1$ (stationary series)

(ii) $y_t = \phi + \rho y_{t-1} + \varepsilon_t$ *and* $\rho = 1$ (non stationary series)

When $\rho = 1$, the deviation of $\hat{\rho}$ from its true value $T(\hat{\rho} - 1)$ has a limiting distribution that is skewed to the left, and in 2/3 of generated samples, the usual estimator in expression 3 will take values such that $(\hat{\rho} - 1) < 0$.

2.3. The Functional Central Limit Theorem

Consider the process $\Delta y_t = \delta_0 + \delta_1 t + (\rho - 1)y_{t-1} + u_t$. When $\rho = 1$, the normalizing factor of a test based directly on the OLS estimator of ρ or based on $(\rho - 1)$, is n rather than $\sqrt{n}$ and standard asymptotic theory does not apply to the statistics. Theorems that involve taking limits in a function space provide the basis for a generalization of the the Central Limit Theorem, which provides a useful asymptotic theory for hypothesis testing in dynamic models. That generalization is the Functional Central Limit Theorem (FCLT) which plays a crucial role in the asymptotic theory of integrated processes.

We can re-write the non stationary process $y_t = y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim iid\ N(0, 1)$, $y_0 = 0$, $t = 1, ..., T$. in the form

$$\begin{aligned} y_t =& \ \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t \\ y_t \sim& \ N(0, t) \end{aligned}$$

To understand the idea behind the FCLT, write the change between t and $t-1$, $\varepsilon_t = y_t - y_{t-1}$ as the sum of distinct sub-periods, for instance $\varepsilon_t = \epsilon_{1t} + \epsilon_{2t}$, $\epsilon_{it} \sim iid\ N(0, \frac{1}{2})$, where $\epsilon_{1t} = y_{t-\frac{1}{2}} - y_{t-1}$ and $\epsilon_{2t} = y_t - y_{t-\frac{1}{2}}$; thus $y_t - y_{t-1} = \epsilon_{1t} + \epsilon_{2t} \sim iid\ N(0, 1)$. Writing ε_t in terms of T sub-periods we have

$$y_t - y_{t-1} = \epsilon_{1t} + \epsilon_{2t} + \cdots + \epsilon_{Tt}, \quad \epsilon_{it} \sim iid\ N\left(0, \frac{1}{T}\right)$$

Scaling $\sum_{t=1}^{T} \epsilon_t$ by $\sqrt{T}$ converges weakly to the standard Wiener process as $T \to \infty$, under some regularity conditions, that is

$$T^{-1/2} \sum_{t=1}^{T} \epsilon_t \longrightarrow_d \sigma W(1)$$

where $W(\cdot)$ is a standard Brownian motion: a continuous-time process, associating each date $t \in [0, 1]$ with the scalar $W(t)$ such that

(i) $W(0) = 0$

(ii) For any dates $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$, the changes $[W(t_2) - W(t_1)]$, $[W(t_3) - W(t_2)]$,...,$[W(t_k) - W(t_{k-1})]$ are independent multivariate Gaussian with $[W(s) - W(t)] \sim N(0, s - t)$

(iii) For any given realization, $W(t)$ is continuous in t with probability 1.

Using the continuous mapping theorem, we can also know many other limiting distributions. One of the most important is the limiting distribution of the OLS estimator $\hat{\rho}$ for $y_t = \rho y_{t-1} + e_t$ when $\rho = 1$

$$T(\hat{\rho} - 1) \longrightarrow_d \frac{\int_0^1 W(r)dW(r)}{\int_0^1 W(r)^2 dr} \tag{6}$$

See [15]. This is known as the Dickey-Fuller distribution because [13] and [18] made the tables of this distribution before the publication of the limiting distribution with Wiener processes.

2.4. Testing for Unit Roots

To carry out statistical inference on the parameters of the process $y_t = \mu + \rho y_{t-1} + e_t$, we estimate $\Delta y_t = \mu + \gamma y_{t-1} + \epsilon_t$ and test $H_0 : \gamma = \rho - 1 = 0$ v.s. $\gamma < 0$. As we do not know the value of the true parameter ρ nor the true data generating process, we perform the hypothesis testing on several specifications (see [9])

$$\begin{array}{rl} i) & \Delta y_t = \gamma y_{t-1} + u_t \\ ii) & \Delta y_t = \delta_0 + \gamma y_{t-1} + u_t \\ iii) & \Delta y_t = \delta_0 + \delta_1 t + \gamma y_{t-1} + u_t \\ iv) & \Delta y_t = \delta_0 + \delta_1 t + \delta_2 t^2 + \gamma y_{t-1} + u_t \end{array} \tag{7}$$

After estimating the value of γ one uses the critical values as presented in [13] and [18], obtained from numerical methods, and compare them with a statistic of the form of the $t - statistic$.

2.5. Testing for Explosive Behavior

We find extensions of the Dickey-Fuller test in [15–17] that allow us to test for explosivity. The specification in a test of explosivity differs from those of a unit root in that we consider window sizes. In the following equation

$$\Delta y_t = \mu_{r_w} + \gamma_{r_w} y_{t-1} + \sum_{j=1}^{k} \psi_{rw}^{j} \Delta y_{t-j} + \epsilon_t \tag{8}$$

the term $r_w = r_2 - r_1$ specifies the start and ending points of a subsample, where r_2 and r_1 are expressed as fractions of T. The parameters to be estimated are μ_{r_w}, γ_{r_w} and $\psi_{rw}^{j}, j = 1, \ldots, k$. The term $\epsilon \sim N(0, \sigma_{r_w}^2)$ is the unobserved random shock. Notice that the usual ADF statistic is $ADF_{r_1}^{r_2} = \hat{\gamma_{r_1, r_2}} / se(\hat{\gamma_{r_1, r_2}})$ specifying $r_1 = 0$ and $r_2 = 1$ or the complete sample. The limiting distribucion of ADF_0^1 under the null hypothesis is $\frac{\int_0^1 WdW}{\left[\int_0^1 W^2\right]^{1/2}}$. We can use the mentioned setup for hypothesis testing by following these two steps: first, calculate ADF_0^1 and define your hypothesis as $H_0 : \gamma_{r_w} = 0$ (unit root) and $H_1 : \gamma_{r_w} > 0$ (explosiveness). Second, compare your ADF_0^1 with the right-tail critical value of its limiting distribution. With $r_2 = 1$ and $r_1 = 0$, it is a test of exuberance over the entire sample period.

When there are several episodes of explosivity, [16] propose to estimate 8 recursively by expanding forward the end point of the subsamples, i.e. by fixing $r_1 = 0$ and expanding r_2 from a minimum value r_0 towards 1. As we calculate several $ADF_0^{r_2}$, we keep the supremum of the $ADF_0^{r_2}$s, which is known as the SADF:

$$SADF(r_0) = \sup_{r_2 \in [r_0, 1]} ADF_0^{r_2} \tag{9}$$

From this, we define $H_0 : \gamma_{r_w} = 0$ and $H_1 : \gamma_{r_w} > 0$ and compare the $SADF$ with the critical value on the right tail of the limiting distribution under the null:

$$SADF(r_0) \longrightarrow_d \sup_{r_2 \in [r_0,1]} \frac{{}_0^{r_2} W dW}{[{}_0^{r_2} W^2]^{1/2}}$$

By expanding $r_2 = 1$, we perform an explosivity test on parts of the sample.

Carrying out the test on every subperiod is also feasible by changing both values of r_2 and r_1. With flexible windows we calculate the Generalized SADF

$$GSADF(r_0) = \sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} ADF_{r_1}^{r_2} \tag{10}$$

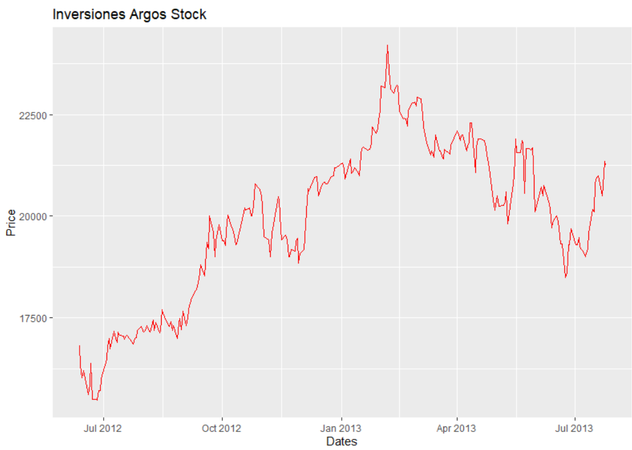
and compare the $GSADF$ with the critical value on the right tail of the limiting distribution:

$$\sup_{r_2 \in [r_0,1], r_1 \in [0, r_2 - r_0]} \frac{(1/2) r_w [W(r_2)^2 - W(r_1)^2 - r_w] - {}_{r_1}^{r_2} W(r) dr [W(r_2) - W(r_1)]}{r_w^{1/2} \left\{ r_w {}_{r_1}^{r_2} W(r)^2 dr - [{}_{r_1}^{r_2} W(r) dr]^2 \right\}^{1/2}}$$

Some recent interesting articles using tests of explosivity include [19–21].

3. Results

I use data on daily prices of Inversiones Argos (Inverargos) stock from 2012-06-12 to 2013-07-25. The OLS estimation of the linear model $y_t = \mu + \rho y_{t-1} + \varepsilon_t$ seems to indicate that the model has a good fit as the $r - squared$ is notably high (97%) and the significance level at which we can reject the null hypotheses of zero mean and zero slope are remarkably small (6,2% and 0.0% respectively; see table 1).



Inversiones Argos stock. Daily closing prices from 2012-06-12 to 2013-07-25.

Figure 4: Inverargos Stock Price

The results of the Dickey-Fuller test from the specifications $i)$ to $iii)$ in expression 7 are in table 2. The test suggests that the series of the stock price is non-stationary which implies that the distribution of the statistic in expression 5 does not follow a t-Student distribution or normal distribution asymptotically and the critical values used in the OLS estimation in table 1 are different to the true critical values.

	Dependent variable:
	y_t
y_{t-1}	0.981
	(0.010)
	t statistic = 93.607
	p-value = 0.000
μ	393.705
	(210.015)
	t statistic = 1.875
	p-value = 0.062
Observations	272
R^2	0.970
Adjusted R^2	0.970
Residual Std. Error	337.765 (df = 270)
F Statistic	8,762.257*** (df = 1; 270)

Note: *p<0.1; **p<0.05; ***p<0.01. Standard errors in parenthesis.

Table 1: OLS Results

	no constant	constant	constant and time trend
DF statistic	0.77	-1.87	-1.83
p-value	0.86	0.36	0.65

Table 2: Dickey-Fuller Test

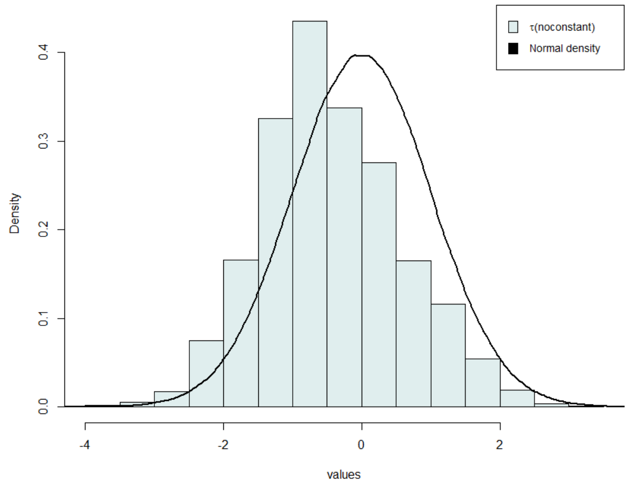
I obtain the critical values that are useful for hypothesis testing with unit roots by simulating 2000 times a time-series with $\rho = 1$, running regressions to estimate γs and calculate the $t - statistic$. From the quantiles of the cumulative distribution of the statistic I record the critical values. The distributions of the $t - statistic$ for three specifications are shown in figure 5. Not only these distributions are not normal but they differ for different specifications of the linear dynamic model as noted by [9].

A visual inspection of the stock price series provides indications of episodes of explosivity. To test for explosivity I calculate the ADF and SADF statistic as well as the GSADF of expression 10 by setting $r_0 = 0.01 + 1.8/\sqrt{T}$ as proposed in [17] and comparing those with the critical values generated from Monte Carlo simulations approximating a Brownian motion.

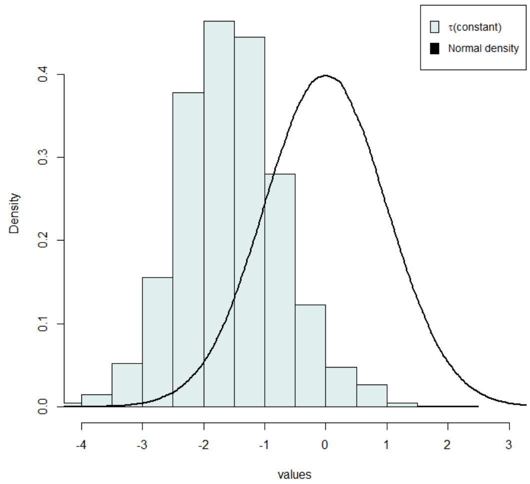
	Statistic	90% c.v.	95% c.v.	99% c.v.
ADF	-1.86	-0.446	-0.0631	0.626
SADF	0.544	1.13	1.45	2.10
GSADF	2.24	1.92	2.16	2.67

Table 3: Tests of Explosive Behavior

The results suggest that there are times during which the series present an exhuberant



a)



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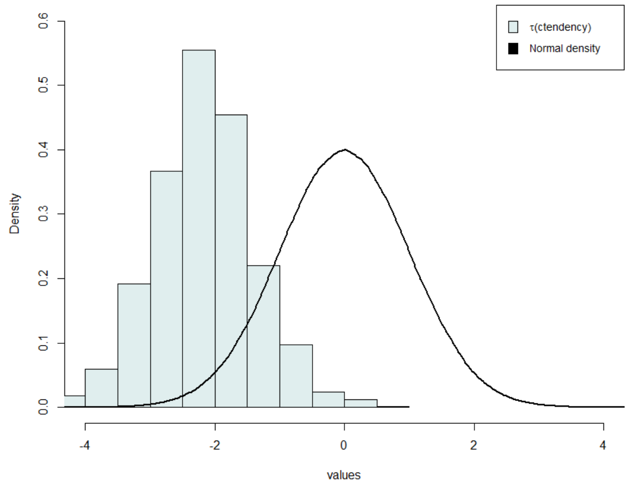
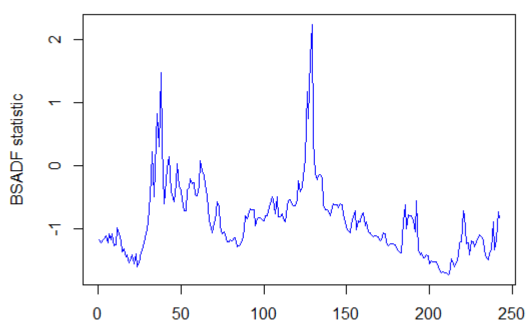


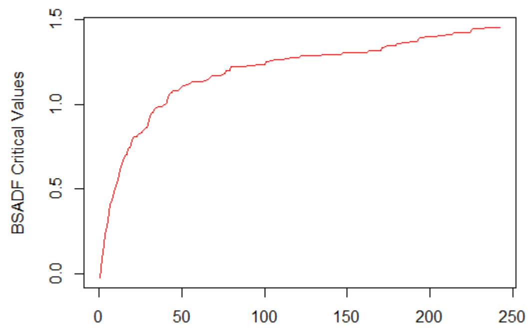
c)

Figure 5: Simulated distributions of the usual $t - test$ for autoregressive processes. Panel a) shows the distribution of the $t-statistic$ for an OLS estimator with $H_0 : \gamma = \rho - 1 = 0$ when the true process is non-stationary and does not have a constant, i.e. $\Delta y_t = (\rho - 1)y_{t-1} + u_t, \rho = 1$. Panel b) shows the distribution when the true process is $\Delta y_t = \delta_0 + (\rho - 1)y_{t-1} + u_t$. Panel c) shows the distribution for the process $\Delta y_t = \delta_0 + \delta_1 t + (\rho - 1)y_{t-1} + u_t$. In the the legends $\tau(noconstant)$ refers to the model without constant; $\tau(constant)$ refers to the model with a constant; $\tau(ctrend)$ is the model with a constant and a trend.

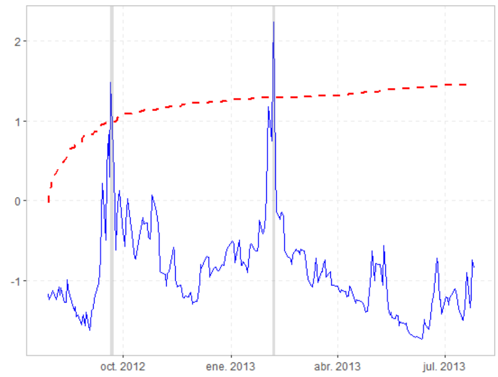
behavior at the 5% level of significance for the GSADF (see table 3). The standard ADF test as well as the SADF fail to reject the null. The superior power of the GSADF follows from the flexible estimation window which makes the GSADF test better for detecting multiple changes in regime or bubbles that burst in-sample. Testing for the dates at which the bubble started and finished is feasible using the Backward SADF test. For each sub-period, I calculate the test and the critical values. The figure 6 panel c) the shaded areas show the time period of the explosivity.



a)



b)



c)

Figure 6: In Date Stamping with the BSADF Test. Panel a) shows the distribution of the values of the BSADF statistic; panel b) presents the critical values at the 5% level of significance; and panel c) shows the dates at which the null hypothesis is rejected.

The dates marked with shaded areas are presented in table 4

Start	Peak	End
2012-09-21	2012-09-21	2012-09-24
2013-02-05	2013-02-06	2013-02-07

Table 4: Date Stamping

4. Discussion

The time-series structure of data allows the researcher to answer important questions that have a dynamic context. Many of those questions are related to the future values of a variable of interest. [22], for example, predict climate phenomena in West Java with rainfall variables and humidity. Also, from time-series data sets, the academic literature has studied the existence of cognitive biases, the effects of lending on agricultural investments and the magnitude of searching costs in financial markets, among others. Specifically in macroeconomics, time-series modeling has been helpful to test the permanent income hypothesis [23, 24], the hypothesis of precautionary savings [25], the persistence of aggregate income in the short term [26], the Deaton (1987) paradox on consumption volatility [27], bubble transmission and formation in the housing market [21] and the predictability of inflation and economic activity from principal component analyses [28].

Pivotal to the methodologies that allow to identify dynamic structures is the functional limit theorem which provides the researcher with the appropriate critical values when the usual $t - Student$ or normal distributions are not suitable for hypothesis testing.

5. Conclusions

In this paper I provide a comprehensive exploration of the use of limiting distributions in empirical scientific research, particularly focusing on the role of the t statistic in hypothesis testing within linear models. Through an examination of various applications spanning fields such as food science, medical research, and economics, the paper elucidates the practical significance of statistical principles like the Law of Large Numbers and the Central Limit Theorem, alongside the Neyman-Pearson principle, in facilitating clear decision-making processes in hypothesis testing.

Moreover, the paper emphasizes the necessity of analyzing the dynamic dependence structure of data, particularly in time-series contexts, before relying solely on the t statistic for hypothesis testing. It underscores the importance of considering the temporal dimension when conducting empirical analyses, especially in fields where sequential interpretation of data is paramount. Furthermore, the discussion on testing for Unit Roots and Explosive Behavior in time-series data highlights the diverse array of statistical tools available to researchers for exploring dynamic structures within their datasets. By presenting examples from various domains, ranging from climate prediction to macroeconomics, the paper underscores the broad applicability of these statistical techniques in addressing a multitude of research questions.

In essence, this paper contributes to the methodological toolkit of researchers by delineating the necessary conditions for proper hypothesis testing within linear models and elucidating asymptotic tools specific to information with time-series properties. It stresses the importance of considering dynamic structures in empirical analyses and provides valuable insights into statistical methodologies that go beyond traditional approaches based on normal or t distributions, thereby enriching the empirical scientific research landscape.

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