Journal de Ciencia e Ingeniería, vol.16, no.1, pp.33-36, 2024
http://jci.uniautonoma.edu.co
DOI: https://doi.org/10.46571/JCI.2024.1.5

Corporación Universitaria Autónoma del Cauca
ISSN 2145-2628, e-ISSN 2539-066X
Mathematics

Möbius transformation and the Riemann sphere

Transformación de Möbius y la esfera de Riemann

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Abstract. We exhibit that the 3-rotations and Lorentz mappings generate Möbius transformations in the complex plane.

Keywords. Euler angles; Cayley-Klein parameters; Lorentz transformations; Riemann sphere; Euler-Rodrigues parameters; Möbius mapping.

Resumen. En este trabajo se muestra que las 3-rotaciones y los mapeos de Lorentz generan transformaciones de Möbius en el plano complejo.

Palabras Claves. Ángulos de Euler, Parámetros de Cayley-Klein, Transformaciones de Lorentz, Esfera de Riemann, Parámetros de Euler-Rodrigues, Mapeo de Möbius.

How cite. J.D. Bulnes, H.E. Caicedo-Ortiz and J. López-Bonilla, Möbius transformation and the Riemann sphere. *Jou. Cie. Ing.*, vol. 16, no. 1, pp. 33-36, 2024. doi:10.46571/JCI.2024.1.5

Received: 21/03/2024 **Reviewed:** 06/02/2024 **Accepted:** 06/16/2024

1. Introduction

The stereographic projection establishes a correspondence between the points of the unit sphere and those of the Argand plane [1–4], thus it is easy to see that:

$$\lambda = X - iY = \frac{x - iy}{1 - z} = \cot\left(\frac{\theta}{2}\right) e^{-i\varphi} \tag{1}$$

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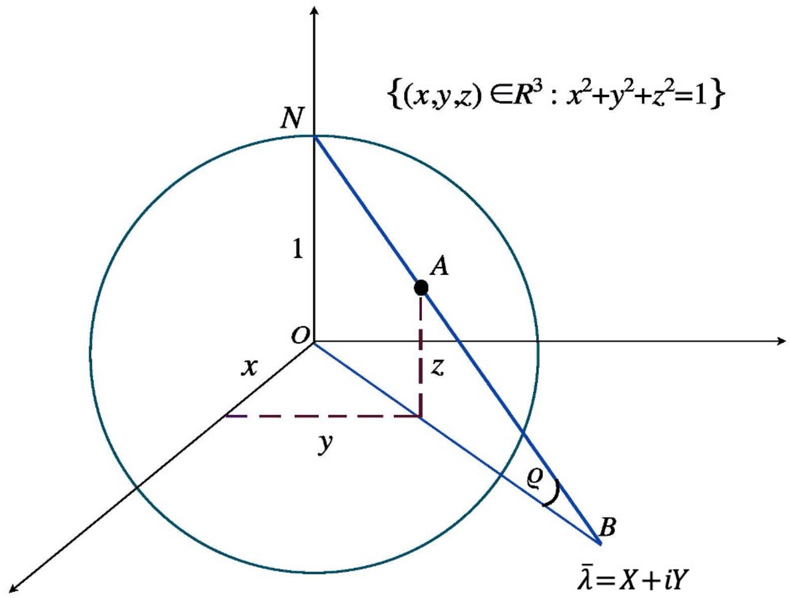

Figure 1: The unit sphere is projected stereographically from its north pole to its equator.

where θ and φ are spherical angles. Under an arbitrary 3-rotation about the origin, each point of the sphere is mapped into another point on the sphere, hence via the stereographic projection a rotation determines a transformation of the complex onto itself. In fact, such three-dimensional rotation is given by [5,6]:

$$\widetilde{x} - i\widetilde{y} = \left(\overline{\alpha}\lambda + \overline{\beta}\right)\left(\overline{\alpha} - \overline{\beta\lambda}\right)(1-z), \qquad 1-\widetilde{z} = (-\beta\lambda+\alpha)\left(\overline{\alpha} - \overline{\beta\lambda}\right)(1-z), \tag{2}$$

where the complex numbers α and β are the Cayley-Klein parameters [7] with the constraint $\alpha\overline{\alpha} + \beta\overline{\beta} = 1$. Therefore:

$$\widetilde{\lambda} = \frac{\widetilde{x} - i\widetilde{y}}{1-\widetilde{z}} = \frac{\overline{\alpha}\lambda + \overline{\beta}}{-\beta\lambda + \alpha,} \tag{3}$$

which is a Möbius mapping [8] first obtained by Gauss [9,10]; then, the most general rotation of the Riemann sphere can be expressed as a Möbius transformation of the form (3). In terms of Euler angles [11] the relation (3) acquires the structure [12]:

$$\widetilde{\lambda} = \frac{e^{-i\varphi} - i\lambda\cot(\frac{\theta}{2})}{\lambda - ie^{-i\varphi}\cot(\frac{\theta}{2})}e^{i\psi} = \frac{(a_0 - ia_3)\,\lambda - (a_2 + ia_1)}{(a_2 - ia_1)\,\lambda + (a_0 + ia_3)}, \qquad a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1, \tag{4}$$

with the participation of the Euler-Rodrigues real parameters [13–15,17].

Now we consider a null cone with vertex in $(0,0,0,0)$, after for some value of $x^0 = ct$ we construct Cartesian axes for the variables $x^{'} = \frac{x}{ct}, y^{'} = \frac{y}{ct}, z^{'} = \frac{z}{ct}$, thus we have a unit Riemann sphere because $(x^{'})^2 + (y^{'})^2(z^{'})^2 = 1$ for a null ray, which can be projected as in the Fig. 1, then:

$$\lambda = \frac{x–iy}{x^0 - z}, \qquad x^2 + y^2 + z^2 = (x^0)^2, \tag{5}$$

and under an arbitrary Lorentz transformation [18, 19]:

$$\widetilde{x} - i\widetilde{y} = \left(\overline{\alpha}\lambda + \overline{\beta}\right)\left[\delta\left(x^0 - z\right) + \gamma\left(x + iy\right)\right], \quad \widetilde{x^0} - \widetilde{z} = \left(\overline{\gamma}\lambda + \overline{\delta}\right)\left[\delta\left(x^0 - z\right) + \gamma\left(x + iy\right)\right], \tag{6}$$

with the constraint $\alpha\delta - \beta\gamma = 1$, we obtain again a Möbius mapping:

$$\widetilde{\lambda} = \frac{\overline{\alpha}\lambda + \overline{\beta}}{\overline{\gamma}\lambda + \overline{\delta}} \tag{7}$$

which has connection with special relativity [20]; Coxeter [21] comments that this connection was observed by Liebmann [22]. Thus, the complex mappings that correspond to Lorentz rotations are the Möbius transformations with 6 degrees of freedom.

Remark 1: In Fig. 1 we have the relations:

$$NA = 2\sin\rho, \qquad NB = \frac{1}{\sin\rho} = \frac{2}{NA} \tag{8}$$

If we take the north pole of the Riemann sphere as centre of a sphere K with radius $\sqrt{2}$, then (8) means that the point B is the inversion in K of the point A. We may remember that the "inversion in a sphere" also is called "transformation by reciprocal radii" [8, 10, 23–25].

Remark 2: The null vector $(x^\mu) = (x^0, x, y, z)$ has associated the Cartan spinor [26]:

$$\left(X^{A\dot{B}}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + z & x + iy \\ x - iy & x^0 - z \end{pmatrix} = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}\begin{pmatrix} \xi^{\dot{1}} & \xi^{\dot{2}} \end{pmatrix} = \begin{pmatrix} \xi^1\xi^{\dot{1}} & \xi^1\xi^{\dot{2}} \\ \xi^2\xi^{\dot{1}} & \xi^2\xi^{\dot{2}} \end{pmatrix}, \tag{9}$$

where $x^0 > 0$ and $x^0 + z = (x + iy)\lambda$, then $\xi^2\xi^{\dot{1}} = \frac{x-iy}{\sqrt{2}}$ and $\xi^2\xi^{\dot{2}} = \frac{x^0-z}{\sqrt{2}}$, therefore:

$$\lambda = \frac{\xi^{\dot{1}}}{\xi^{\dot{2}}}, \tag{10}$$

that is, the stereographic projection is the quotient of the complex components of spinor $\xi^{\dot{A}}$, in other words, $\xi^{\dot{1}}$ and $\xi^{\dot{2}}$ are homogeneous coordinates of λ [10, 27]. Hence a pair of projective scalar $\xi^{\dot{1}}$ and $\xi^{\dot{2}}$ determine a unique point of the light-cone at each point of space-time; each value of the ratio $\xi^{\dot{1}}/\xi^{\dot{2}}$ determines a line of the cone. The homogeneous coordinates were invented by Möbius (1827) and Plücker (1830) [28].

Remark 3: The Riemann sphere [29] permits a geometric interpretation of spin states for the electron and photon [30–32].

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