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Combinatorial

Sun's Binomial Inversion Formulae, Euler's Operator, and Z-Transform

Fórmulas de Inversión Binomial de Sun, Operador de Euler y Transformada Z

R. Sivaraman*[1], J. D. Bulnes+[2], Laid Elkhiri−[3], J. López-Bonilla#[4]
*** Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College, Chennai 600 106, Tamil Nadu, India**
+Departamento de Ciencias Exatas e Tecnología, Universidade Federal do Amapá, Rod. Juscelino Kubitschek, Jardin Marco Zero, 68903-419, Macapá, AP, Brasil
−Department of Material Sciences, RECITS Laboratory, Tiaret University, Algeria
#ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract. We exhibit the participation of Euler's operator in the Sun's binomial inversion formulae and we use the Z-transform to give elementary proofs of such formulae.

Keywords. Binomial inversion formula; Euler's operator; Z-transform.

Resumen. Mostramos la participación del operador de Euler en las fórmulas de inversión binomial de Sun y utilizamos la transformada Z para proporcionar pruebas elementales de dichas fórmulas. .

Palabras Claves. Fórmula de inversión binomial; Operador de Euler; Transformada Z.

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[1] *e-mail: rsivaraman1729@yahoo.co.in*
[2] *e-mail: bulnes@unifap.br*
[3] *e-mail: laid@univ-tiaret.dz*
[4] *e-mail: jlopezb@ipn.mx*

1. Introduction

The generating function for the Bernoulli numbers is given by [1–4]:

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \tag{1}$$

which, for $t = \frac{1}{z}$, gives the following Z-transform [5–7] for the sequence $\{B_0 0!, B_1 1!, B_2 2!, \ldots\}$:

$$\frac{1}{z}\left(e^{1/z} - 1\right)^{-1} = Z\left\{B_n \frac{n!}{n!}\right\} \tag{2}$$

In Sec. 2, we determine the sequence $Q_n \frac{n!}{n!}$ such that:

$$\left(e^{1/z} - \lambda\right)^{-1} = Z\left\{Q_n \frac{n!}{n!}\right\}, \quad \lambda \neq 1 \tag{3}$$

In Sec. 3, we use these relations to deduce Sun's binomial inversion formulae [8–10].

The Sec. 4 considers the case $\lambda = -1$, and it is proven that the corresponding Q_k can be written in terms of Bernoulli numbers. Sec. 5, for $\lambda \neq 1$, shows the connection between Q_m and Euler's operator [11, 12] via Grünert's operational formula [2, 12, 13].

2. Construction of the quantities Q_n with the property (3)

It is evident from the identity:

$$\frac{1}{z}\left(e^{1/z} - 1\right)^{-1} - \frac{1}{z}\left(e^{1/z} - \lambda\right)^{-1} = (1-\lambda)\frac{1}{z}\left(e^{1/z} - 1\right)^{-1} \cdot \left(e^{1/z} - \lambda\right)^{-1} \tag{4}$$

where we can apply Z^{-1} to (2) and (3) to deduce the recurrence relation:

$$B_n - nQ_{n-1} = (1-\lambda)\sum_{k=0}^{n}\binom{n}{k} Q_k B_{n-k}, \quad n \geq 0 \tag{5}$$

For $n = 0$, this implies the value:$Q_0 = (1-\lambda)^{-1}$. The expression (5) is equivalent to:

$$nQ_{n-1} = -(1-\lambda)\sum_{k=1}^{n}\binom{n}{k} Q_k B_{n-k}, \quad n \geq 1 \tag{6}$$

This allows determining the quantities Q_m. For example:

$$Q_1 = -\frac{1}{1-\lambda^2}, \quad Q_2 = \frac{1+\lambda}{1-\lambda^3}, \quad Q_3 = -\frac{\lambda^2 + 4\lambda + 1}{1-\lambda^4}, \ldots \tag{7}$$

If we recall the property [1]:

$$\frac{n}{j} S_{n-1}^{[j-1]} = \sum_{k=j}^{n}\binom{n}{k} S_k^{[j]} B_{n-k}, \quad 1 \leq j \leq n \tag{8}$$

involving the Stirling numbers of the second kind [1–3], then it is easy to obtain the following explicit solution of (6):

$$Q_n = \sum_{j=0}^{n} \frac{(-1)^j j!}{(1-\lambda)^{j+1}} S_n^{[j]}, \quad n \geq 0, \lambda \neq 1, \tag{9}$$

this is in agreement with the values of Q_1, Q_2, and Q_3 obtained earlier. Hence, from (3) and (9):

$$Z\left\{\frac{1}{n!}\sum_{j=0}^{n} \frac{(-1)^j j!}{(1-\lambda)^{j+1}} S_n^{[j]}\right\} = \left(e^{1/z} - \lambda\right)^{-1} \tag{10}$$

3. Sun's binomial inversion formulae

If we consider the binomial expression:

$$F_n = \sum_{k=0}^{n} \binom{n}{k} f_k - \lambda f_n, \quad n \geq 0 \tag{11}$$

then Sun [8–10] obtained the corresponding inversion formulae:

$$f_n = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} F_k B_{n+1-k}, \quad \lambda = 1 \tag{12}$$

$$f_n = -\sum_{m=0}^{n} \binom{n}{m} F_m \sum_{k=0}^{n-m} \frac{k!}{(1-\lambda)^{k+1}} S_{n-m}[k], \quad \lambda \neq 1; \tag{13}$$

with the results deduced in Sec. 2 we can give elementary proofs of (12) and (13) in fact from (11) for the case $\lambda = 1$:

$$F_n = \sum_{k=0}^{n} \binom{n}{k} f_k - f_n \quad \therefore \quad F_0 = 0 \tag{14}$$

where we apply the Z-transform to obtain:

$$Z\left\{\frac{F(n)}{n!}\right\} = Z\left\{\frac{F(n)}{n!}\right\} Z\left\{\frac{1}{n!} - 1\right\} \quad = Z\left\{\frac{1}{n!}\right\} = e^{1/z} \tag{15}$$

then:

$$\mathcal{Z}\left\{\frac{f(n)}{n!}\right\} = \left(e^{\frac{1}{z}} - 1\right)^{-1} \mathcal{Z}\left\{\frac{F(n)}{n!}\right\} \underset{=\!=}{\underline{(2)}} z\mathcal{Z}\left\{\frac{B_n}{n!}\right\} \mathcal{Z}\left\{\frac{F(n)}{n!}\right\}, \tag{16}$$

such that:

$$zZ\left\{\frac{F(n)}{n!}\right\} = z\left(\frac{F_1}{1!z} + \frac{F_2}{2!z^2} + \frac{F_3}{3!z^3} + \dots\right) = Z\left\{\frac{F(n+1)}{(n+1)!}\right\}, \tag{17}$$

hence (16) implies the binomial transform: $f(n)n! = \sum_{k=0}^{n} \frac{F(k+1)}{(k+1)! \frac{B_{n-k}}{(n-k)!}}$

which is equivalent to (12), q.e.d.

The Z-transform of (11) for the case $\lambda \neq 1$ gives the relation:

$$Z\left\{\frac{F(n)}{n!}\right\} = Z\{f_n n!\} Z\left\{\frac{1}{n!} - \lambda\right\} = e^{1/z-\lambda} Z\{f_n n!\}$$

therefore:

$$Z\{f(n)n!\} = \left(e^{1/z-\lambda}\right)^{-1} Z\{F_n n!\}$$

which implies (13) *q.e.d.*

Remark 1
We can exhibit an alternative proof of (12). In fact, we know the relation [14]:

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k} = \binom{n}{m}\left(B_{n-m} + \delta_{n-m,1}\right), \quad n \geq m \geq 0 \tag{18}$$

Thus:

$$\sum_{m=0}^{n}\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k} f(m) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} f(m) + \binom{n}{n-1} f(n-1)$$

Therefore:

$$nf(n-1) = \sum_{k=0}^{n} \binom{n}{k}\left[\sum_{m=0}^{k}\binom{k}{m} f(m) - f(k)\right] B_{n-k},$$

in agreement with (11) and (12) for $\lambda = 1$.

4. Q_n for $\lambda = -1$
From (11) and (13) with $\lambda = -1$:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) + f(n), \quad Q_n = \sum_{j=0}^{n} \frac{(-1)^j j!}{2^{j+1}} S_n^{[j]}, \tag{19}$$

besides we know the relation [2, 15, 16]:

$$B_{n+1} = \frac{n+1}{1-2^{n+1}} \sum_{j=0}^{n} \frac{(-1)^j j!}{2^{j+1}} S_n^{[j]} \tag{20}$$

therefore:

$$Q_n = \frac{1-2^{n+1}}{n+1} B_{n+1}, \quad Q_{2m} = 0, \quad m \geq 1, \quad f(n) = \sum_{j=0}^{n} \binom{n}{j} \frac{1-2^{n+1-j}}{n+1-j} F(j) B_{n+1-j}, \tag{21}$$

which gives the following Z-transform [17]:

$$Z\left\{\frac{1-2^{n+1}}{(n+1)!} B_{n+1}\right\} = \left(e^{1/z}+1\right)^{-1}. \tag{22}$$

5. Q_n in terms of the Euler operator for $\lambda \neq 1$
Grünert's formula [2, 12, 13] is given by:

$$\left(x\frac{d}{dx}\right)^n g(x) = \sum_{j=0}^{n} x^j S_n^{[j]} g^{(j)}(x) \tag{23}$$

for the Euler operator $x\frac{d}{dx}$ [11, 12], which can be applied to the function:

$$g(x) = \frac{1}{x-\lambda}, \quad \therefore \quad g^{(j)}(1) = \frac{(-1)^j j!}{(1-\lambda)^{j+1}}, \tag{24}$$

Then (13), (23), and (24) imply the interesting relationship:

$$Q_n = \sum_{j=0}^{n} S_n^{[j]} g^{(j)}(1) = \left[\left(x\frac{d}{dx}\right)^n \frac{1}{x-\lambda}\right]_{x=1} \tag{25}$$

Thus, the first few values of Q_n are:

$$Q_0 = \frac{1}{1-\lambda}, \quad Q_1 = -\frac{1}{(1-\lambda)^2}, \quad Q_2 = \frac{1+\lambda}{(1-\lambda)^3}, \quad Q_3 = -\frac{\lambda^2+4\lambda+1}{(1-\lambda)^4}, \ldots \tag{26}$$

Remark 2:
The inversion of (19) is given by:

$$(-1)^n \frac{n!}{(1-\lambda)^{n+1}} = \sum_{j=0}^{n} Q_j S_n^{(j)} \tag{27}$$

in terms of Stirling numbers of the first kind [1, 2].

Remark 3
The inversion of (20) generates the Shirai-Sato identity [16, 18]:

$$\sum_{j=0}^{n} \frac{1-2^{j+1}}{j+1} B_{j+1} S_n^{(j)} = \frac{(-1)^n\, n!}{2^{n+1}}, \quad n \geq 0. \tag{28}$$

Remark 4
From (21) and (25):

$$B_{n+1} = \frac{n+1}{1-2^{n+1}} \left[\left(x \frac{d}{dx} \right)^n \frac{1}{x+1} \right]_{x=1}. \tag{29}$$

Remark 5
The expression (11) is a binomial transform [19] if $\lambda = -1$, then (25) implies that $Q_n = (-1)^n$ and thus (13) gives the well-known inversion property [2]:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) \iff f(n) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F(k). \tag{30}$$

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