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On a coefficient identity for powers of Taylor series

Sobre una identidad de coeficientes para potencias de series de Taylor

J.D. Bulnes[+1] **M.A.I. Travassos**[+2] **and J. López-Bonilla**[**3]
[+]**Dep. Ciências Exatas e Tecnologia, Universidade Federal do Amapá, Macapá, 68903-419, AP, Brasil**
[**]**ESIME-Zacatenco, Instituto Politécnico Nacional, Ciudad de México, México**

Abstract. We employ the Z-transform to obtain a recurrence relation to determine the coefficients g_n in $\left(\sum_{k=0}^{\infty} f_k x^k\right)^p = \sum_{n=0}^{\infty} g_n x^n$, in terms of the quantities f_k, where p is an arbitrary real or complex number.

Keywords. Z-transform; powers of Taylor series.

Resumen. Utilizamos la transformada Z para obtener una relación de recurrencia que determina los coeficientes g_k en $\left(\sum_{n=0}^{\infty} f_n x^n\right)^p = \sum_{n=0}^{\infty} g_n x^n$ en términos de las cantidades f_m, donde p es un número real o complejo.

Palabras Claves. Transformada Z; potencias de series de Taylor.

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1. Introduction

Gould [1] studied the relation:

$$\left(\sum_{k=0}^{\infty} f_k x^k\right)^p = \sum_{n=0}^{\infty} g_n x^n, \tag{1}$$

where p is any real or complex number, obtaining the recurrence relation:

$$\sum_{k=0}^{n} \left[k(p+1) - n\right] f_k g_{n-k} = 0, \quad n \geq 0, \tag{2}$$

[1] *e-mail: bulnes@unifap.br*
[2] *e-mail: angelicaptravass@gmail.com*
[3] *e-mail: jlopezb@ipn.mx*

which allows determining the coefficients g_n in terms of the quantities f_k.

In Section 2, we employ the Z-transform [2,3] to give a simple deduction of this recurrence relation.

2. Z-transform Approach to Powers of Taylor Series

The expression (1) above, with $x = \frac{1}{z}$, gives the following relation between the corresponding Z-transforms of the sequences $\{f_k\}$ and $\{g_n\}$:

$$(F(z))^p = G(z), \tag{3}$$

where we apply logarithmic differentiation to obtain:

$$p\,G \cdot \left(-z\frac{d}{dz}F\right) = F \cdot \left(-z\frac{d}{dz}G\right),$$

which is the product of Z-transforms:

$$p \cdot \mathcal{Z}\{g_n\} \cdot \mathcal{Z}\{mf_m\} = \mathcal{Z}\{f_m\} \cdot \mathcal{Z}\{mg_m\},$$

implying the convolution:

$$p\sum_{k=0}^{n} kf_k g_{n-k} = \sum_{k=0}^{n} f_k \cdot (n-k)g_{n-k},$$

in complete agreement with (2), *q.e.d.*

The relation (3) indicates an alternative manner to determine the coefficients g_n, in fact, first we construct $F(z)$ for the sequence $\{f_k\}$, which gives $G(z)$ via (3), and finally we apply Z-transform $\mathcal{Z}^{-1}$ [2–4] to obtain g_j. For example, if $f_m = 1$ for $m \geq 0$, then $F(z) = (1-z)^{-1}$, and therefore:

$$G(z) = \frac{z^p}{(z-1)^p} \xrightarrow{\mathcal{Z}^{-1}} g_n = \binom{n+p-1}{p-1},$$

that is:

$$\left(\sum_{n=0}^{\infty} x^n\right)^p = \sum_{n=0}^{\infty} \binom{n+p-1}{p-1} x^n. \tag{4}$$

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