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Special Functions

Some congruences around Fermat quotients

Algunas congruencias relacionadas con los cocientes de Fermat

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Abstract. In this paper, we give some interesting congruences involving harmonic numbers and sequences in the form $W_n = xa^n + yb^n$, related to Fermat quotients, such as $\sum_{n=1}^{m} W_n H_n \equiv f \pmod p$, for the cases $m = p-1$ and $m = \frac{p-1}{2}$, where $q_p(a) = \frac{a^{p-1}-1}{p}$, $a, b \in \mathbb{Z} - p\mathbb{Z}$.

Keywords. Harmonic numbers; Fermat quotients.

Resumen. En este artículo, presentamos algunas congruencias interesantes que involucran números armónicos y secuencias en la forma $W_n = xa^n + yb^n$, relacionadas con los cocientes de Fermat, como $\sum_{n=1}^{m} W_n H_n \equiv f \pmod p$, para los casos $m = p-1$ y $m = \frac{p-1}{2}$, donde $q_p(a) = \frac{a^{p-1}-1}{p}$, $a, b \in \mathbb{Z} - p\mathbb{Z}$.

Palabras Claves. Números armónicos; Cocientes de Fermat.

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1. Introduction

Pierre de Fermat first stated the following result in a letter dated October 18, 1640, to his friend and confidant Frénicle de Bessy:

"For all prime p and $a \in \mathbb{Z} - p\mathbb{Z}$ we have $a^{p-1} \equiv 1 \pmod p$."

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The condition $a \in \mathbb{Z} - p\mathbb{Z}$ is equivalent to $(a, p) = 1$. It follows from Fermat's small theorem that

$$q_p(a) = \frac{a^{p-1} - 1}{p},$$

where $q_p(a) \in \mathbb{Z}$. Fermat quotients have been of interest since the time of Eisenstein and Glaisher in the nineteenth century. Glaisher [3] proved that for a prime $p \geq 3$:

$$\sum_{n=1}^{p-1} n^{p-1} \equiv -2q_p(2) \pmod{p}. \tag{1}$$

On the other hand, Guo-Chu [4] defined generalized harmonic numbers by:

$$H_\alpha(n) = \sum_{k=1}^{n} \frac{\alpha^k}{k}, \quad n \in \mathbb{N},$$

where α is an appropriate parameter, and with $\alpha = 1$ we have the harmonic numbers [1].

Wolstenholme [7] discovered that for any prime number $p \geq 5$:

$$H_{p-1} \equiv 0 \pmod{p^2}.$$

The Binet formulas for the sequences $\{U_n\}$ and $\{V_n\}$ are given by:

$$U_n = \frac{a^n - b^n}{a - b}, \quad V_n = a^n + b^n, \tag{2}$$

where a and b are the roots of the equation $x^2 - rx - 1 = 0$; when $r = 1$, $U_n = F_n$ (the n-th Fibonacci number) and $V_n = L_n$ (the n-th Lucas number).

Granville [2] showed that for an arbitrary prime number $p \geq 5$:

$$H_{p-1}(a) \equiv \frac{(a-1)^{p-1} - a^p + 1}{p} \pmod{p},$$

and Koparal-Omur [5] gave the following congruence:

$$H_{\frac{p-1}{2}}(a) \equiv \frac{p}{2} - \frac{(\sqrt{a} + 1)^p - (\sqrt{a} - 1)^p}{p} \pmod{p}.$$

In light of this important work, here we study some congruences involving harmonic numbers and related to Fermat quotients.

2. Some congruences related to Fermat quotients

Firstly we give another proof of Glaisher's congruence modulo p.

Theorem 1. For any prime $p > 3$ and $a \in \mathbb{Z} - p\mathbb{Z}$, $a \neq 1$, we have:

$$H_{p-1}(a) \equiv -aq_p(a) + (a-1)q_p(1-a) \pmod{p}. \tag{3}$$

Proof. From the expression:

$$a^{p-1}H_{p-1} = \sum_{k=0}^{p-1} a^k H_k - \sum_{k=0}^{p-2} a^k H_k = \sum_{k=0}^{p-2} a^{k+1} H_{k+1} - \sum_{k=0}^{p-2} a^k H_k,$$

$$= (a-1)\sum_{k=0}^{p-2} a^k H_k + \sum_{k=0}^{p-2} \frac{a^{k+1}}{k+1}.$$

Since $H_{p-1} \equiv 0 \pmod p$, we obtain the relation:

$$\sum_{k=0}^{p-1} a^k H_k = \frac{H_{p-1}(a)}{1-a} \pmod p, \tag{4}$$

besides, for any $k \in \{1, \cdots, p-1\}$, clearly: $\binom{p-1}{k} \equiv (-1)^k(1-pH_k) \pmod{p^2}$, then:

$$\frac{(1-a)^{p-1}}{p} = \frac{1}{p}\sum_{k=0}^{p-1}\binom{p-1}{k}(-a)^k \equiv \frac{1}{p} + \frac{1}{p}\sum_{k=1}^{p-1}(-1)^k(-a)^k(1-pH_k) \pmod p,$$

$$= \frac{1}{p} + \frac{1}{p}\sum_{k=1}^{p-1} a^k - pa^k H_k \pmod p = \frac{1}{p} + \frac{1}{p} a \frac{a^{p-1}-1}{a-1} - \sum_{k=1}^{p-1} a^k H_k \pmod p.$$

Thus:

$$\frac{(1-a)^{p-1}-1}{p} - \frac{a}{a-1}\frac{a^{p-1}-1}{p} = -\sum_{k=1}^{p-1} a^k H_k \pmod p, \tag{5}$$

from (4) and (5), the proof of (3) is complete.

In particular, since $q_p(\pm 1) = 0$, then for $a = 2$, appears the Glaisher's congruence (1).

Theorem 2. For any prime $p \geq 3$ and $ab \in \mathbb{Z} - p\mathbb{Z}$, $a, b \neq 1$, we have:

$$q_p(a) + q_p(b) \equiv q_p(ab) \pmod p, \tag{6}$$

$$q_p(a) - q_p(b) \equiv q_p\left(\frac{a}{b}\right) \pmod p. \tag{7}$$

Proof. According to the very definition of Fermat quotients:

$$q_p(ab) = \frac{a^{p-1}b^{p-1}-1}{p} = \frac{(1+pq_p(a))(1+pq_p(b))-1}{p}.$$

$$q_p(ab) = q_p(a) + q_p(b) + pq_p(a)q_p(b) \equiv q_p(a) + q_p(b) \pmod p,$$

and similarly:

$$q_p\left(\frac{1}{a}\right) = \frac{\left(\frac{1}{a}\right)^{p-1}-1}{p} = \frac{1-a^{p-1}}{pa^{p-1}} = -q_p(a) \pmod p.$$

Thus, the proof of (7) is complete.

Theorem 3. For any prime $p > 3$ and $a, b \in \mathbb{Z} - p\mathbb{Z}$, $a, b \neq 1 \pmod p$, we have:

$$\sum_{k=1}^{p-1} W_k H_k = \frac{q_p(a^{b-1}b^{a-1})}{(a-1)(b-1)} + q_p\left(\frac{ab}{(1-a)(1-b)}\right) \pmod p, \tag{8}$$

and:

$$(a-b)\sum_{k=1}^{p-1} U_k H_k = \frac{q_p(\frac{a^{b-1}}{b^{a-1}})}{(a-1)(b-1)} + q_p\left(\frac{a(1-b)}{(1-a)b}\right) \pmod p. \tag{9}$$

Proof. From the expression (5):

$$\sum_{k=0}^{p-1} a^k H_k = \frac{1}{1-a}\left(-aq_p(a) + (a-1)q_p(1-a)\right),$$

$$= \frac{a}{1-a}q_p(a) - q_p(1-a) = \frac{1}{a-1}q_p(a) + q_p\left(\frac{a}{1-a}\right).$$

therefore:

$$\sum_{k=1}^{p-1} W_k H_k = \frac{1}{a-1}q_p(a) + q_p\left(\frac{a}{1-a}\right) + \frac{1}{b-1}q_p(b) + q_p\left(\frac{b}{1-b}\right),$$

$$= \frac{1}{a-1}q_p(a) + \frac{1}{b-1}q_p(b) + q_p\left(\frac{ab}{(1-a)(1-b)}\right),$$

$$= \frac{q_p(a^{b-1}b^{a-1})}{(a-1)(b-1)} + q_p\left(\frac{ab}{(1-a)(1-b)}\right).$$

Similarly:

$$(a-b)\sum_{k=0}^{p-1} U_k H_k = \frac{1}{a-1}q_p(a) + q_p\left(\frac{a}{1-a}\right) - \left(\frac{1}{b-1}q_p(b) + q_p\left(\frac{b}{1-b}\right)\right),$$

$$= \frac{1}{a-1}q_p(a) - \frac{1}{b-1}q_p(b) + q_p\left(\frac{a(1-b)}{(1-a)b}\right),$$

$$= q_p\left(\frac{a^{b-1}b^{a-1}}{(a-1)(b-1)}\right) + q_p\left(\frac{a(1-b)}{(1-a)b}\right),$$

thus, the proofs of (8) and (9) are complete.

Corollary 1. Since $1-a=b$ and $ab=1$, we have the results:

$$\sum_{k=0}^{p-1} F_k H_k = \frac{1}{a-1}q_p(a) - \frac{1}{a}q_p(a-1) \pmod p,$$

$$\sum_{k=0}^{p-1} L_k H_k = \frac{1}{a-1}q_p(a) - \frac{1}{a}q_p(1-a) \pmod p.$$

Lemma 1. For any prime $p \geq 3$ and $a \in \mathbb{Z} - p\mathbb{Z}$, $a \neq 1 \pmod p$, are valid the relations:

$$H_{\frac{p-1}{2}}(a) \equiv (a-1)q_p(a-1) - (a+1)q_p(a+1) \pmod p, \tag{10}$$

$$\sum_{k=0}^{\frac{p-3}{2}} \frac{a^{2k}}{2k+1} \equiv \frac{1}{2}q_p(a^2-1) + \frac{1}{2a}q_p\left(\frac{a+1}{a-1}\right) - q_p(a) \pmod p. \tag{11}$$

Proof.

1. By the observation:

$$H_{p-1}(a) + H_{p-1}(-a) = H_{\frac{p-1}{2}}(a^2),$$

and the congruence (3), the proof of (10) is complete.

2. From the property:

$$\sum_{k=1}^{\frac{p-3}{2}} \frac{a^{2k+1}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-3}{2}} \frac{a^{2k}}{k} = \frac{a^{p-1}}{p-1} + \sum_{k=1}^{p-1} \frac{a^k}{k},$$

and the congruences (3), (10), and:

$$\frac{1}{p-1} \equiv -1 \pmod{p},$$

the proof of (11) is complete.

Theorem 4. For any prime $p > 3$, $a \in \mathbb{Z} - p\mathbb{Z}$, $|a| \neq 1$, and m a non-negative integer, we deduce the results:

$$\sum_{k=0}^{\frac{p-3}{2}} P_{k,m}(a) H_{2k} \equiv a^2 \sum_{k=0}^{\frac{p-3}{2}} \frac{(k+1)^m a^{2k}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}} k^{m-1} a^{2k} \pmod{p^2}, \tag{12}$$

$$\sum_{k=0}^{\frac{p-3}{2}} (-1)^k Q_{k,m}(a) H_{2k} \equiv a^2 \sum_{k=0}^{\frac{p-3}{2}} \frac{(-1)^k (k+1)^m a^{2k}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}} (-1)^k k^{m-1} a^{2k} \pmod{p^2}, \tag{13}$$

where:

$$P_{k,m}(a) = \left(k^m - a^2(k+1)^m\right) a^{2k}, \quad Q_{k,m}(a) = \left(k^m + a^2(k+1)^m\right) a^{2k}.$$

Proof. We have the relations:

$$\sum_{k=0}^{\frac{p-3}{2}} k^m a^{2k} H_{2k} + \left(\frac{p-3}{2}+1\right)^m a^{p-1} H_{p-1} = \sum_{k=1}^{\frac{p-3}{2}+1} k^m a^{2k} H_{2k},$$

$$= \sum_{k=0}^{\frac{p-3}{2}} (k+1)^m a^{2k+2} H_{2k+2} = a^2 \sum_{k=0}^{\frac{p-3}{2}} (k+1)^m a^{2k} \left(H_{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}\right),$$

$$= a^2 \sum_{k=0}^{\frac{p-3}{2}} (k+1)^m a^{2k} H_{2k} + a^2 \sum_{k=0}^{\frac{p-3}{2}} \frac{(k+1)^m a^{2k}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}} k^{m-1} a^{2k}.$$

Since $H_{p-1} \equiv 0 \pmod{p}$, the proof is complete.

Similarly:

$$\sum_{k=0}^{\frac{p-3}{2}} (-1)^k k^m a^{2k} H_{2k} + (-1)^{\frac{p-1}{2}} \frac{p-1}{2} \left(\frac{p-1}{2}\right)^m a^{p-1} H_{p-1} = \sum_{k=1}^{\frac{p-3}{2}+1} (-1)^k k^m a^{2k} H_{2k},$$

$$= \sum_{k=0}^{\frac{p-3}{2}} (-1)^{k+1} (k+1)^m a^{2k+2} H_{2k+2},$$

$$= a^2 \sum_{k=0}^{\frac{p-3}{2}} (-1)^{k+1} (k+1)^m a^{2k} \left(H_{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}\right).$$

$$= a^2\sum_{k=0}^{\frac{p-3}{2}}(-1)^{k+1}(k+1)^m a^{2k}H_{2k} + a^2\sum_{k=0}^{\frac{p-3}{2}}\frac{(-1)^{k+1}(k+1)^m a^{2k}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}}(-1)^k k^{m-1}a^{2k}.$$

Then, by the Wolstenholme's congruence $H_{p-1} \equiv 0 \pmod{p^2}$, the proof of (13) is complete. In the light of (12) and (13), we get the following examples:

Example 1. *Case* $m = 0$. For any prime $p \geq 3$ and $a \in \mathbb{Z} - p\mathbb{Z}$, $|a| \neq 1 \pmod p$:

$$\sum_{k=0}^{\frac{p-3}{2}} a^{2k}H_{2k} \equiv \frac{a^2 q_p(a)}{a^2-1} - \frac{1}{2}q_p(a^2-1) \pmod p,$$

and:

$$\equiv a^2\sum_{k=0}^{\frac{p-3}{2}}\frac{(-1)^{k+1}a^{2k}}{2k+1} + \frac{1}{2}\sum_{k=1}^{\frac{p-1}{2}}\frac{(-1)^k a^{2k}}{k} \pmod{p^2}.$$

Example 2. *Case* $m = 1$. For any prime $p \geq 3$ and $a \in \mathbb{Z} - p\mathbb{Z}$, $|a| \neq 1 \pmod p$:

$$(1-a^2)\sum_{k=0}^{\frac{p-3}{2}} k a^{2k}H_{2k} \equiv a^2\left(\frac{(a^2+1)q_p(a)}{2(a^2-1)} - \frac{1}{4}q_p(a^2-1) + \frac{1}{4a}q_p\left(\frac{a+1}{a-1}\right)\right) \pmod p.$$

Theorem 5. For any prime $p \geq 3$ and $a, b \in \mathbb{Z} - p\mathbb{Z}$, $|a|, |b| \neq 1 \pmod p$:

$$\sum_{k=0}^{\frac{p-3}{2}} V_{2k}H_{2k} \equiv \frac{a^2 q_p(a)}{a^2-1} + \frac{b^2 q_p(b)}{b^2-1} - \frac{1}{2}q_p\left((a^2-1)(b^2-1)\right) \pmod p,$$

$$(a-b)\sum_{k=0}^{\frac{p-3}{2}} U_{2k}H_{2k} \equiv \frac{a^2 q_p(a)}{a^2-1} - \frac{b^2 q_p(b)}{b^2-1} + \frac{1}{2}q_p\left(\frac{b^2-1}{a^2-1}\right) \pmod p.$$

Corollary 2. Since $1 - a = b$ and $ab = 1$, we have:

$$\sum_{k=0}^{\frac{p-3}{2}} L_{2k}H_{2k} \equiv \frac{a^2+1}{2(a^2-1)}q_p(a) + \frac{a^2-2a+2}{2a(a-2)}q_p(a-1) - \frac{1}{2}q_p(a^2-a-2) \pmod p.$$

$$\sum_{k=0}^{\frac{p-3}{2}} F_{2k}H_{2k} \equiv \frac{3a^2-1}{2(2a-1)(a^2-1)}q_p(a) - \frac{3a^2-6a+2}{2a(2a-1)(a-2)}q_p(1-a) + \frac{1}{2(2a-1)}q_p\left(\frac{a-2}{a+1}\right) \pmod p.$$

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